Wonderful World of Pedal Curves

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My talk will cover

- **Pedal Curves**
- **Negative Pedal Curves**
- **Contra Pedal Curves**
- **Pedal of a Pedal of a Pedal …**

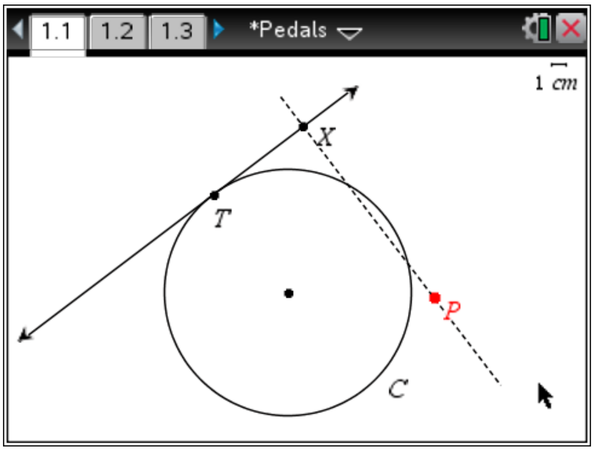
- **Pedal Curves in Space**
- **Pedal Surfaces**

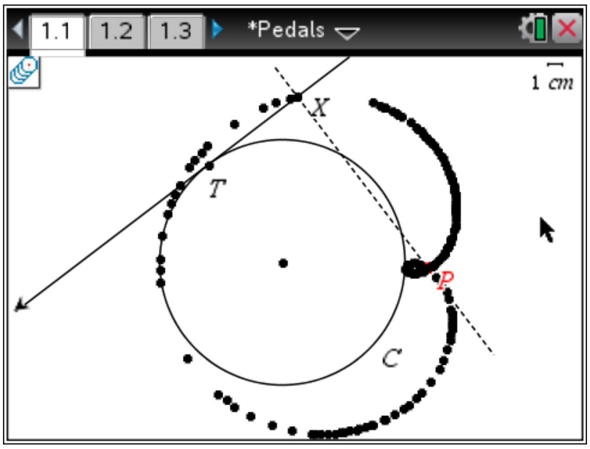
1 What is a Pedal Curve?

For a plane curve *C* and a given fixed point *P* (the pedal point) the **pedal curve of *C*** is the locus of all points *X* with line *PX* being perpendicular to the tangent *t* in *X* to the curve *C*. (Famous mathematicians like McLaurin, Terquem, Quetelet, … investigated pedal curves.)

Let's demonstrate this with TI-NspireCAS:

We will find the pedal curve of a circle.





You can see the construction of point *X*. We drag point *T* along the circle and plot the trace of point *X*. The trace reminds on a Snail of Pascal. We will do the calculation using DERIVE.

We assume that curve *C* is given in parameter form [*u*(*t*), *v*(*t*)].

#2: $[u(t) :=, v(t) :=]$

#3: $\text{tangent} := yp - v(t) = \frac{v'(t)}{u'(t)}\cdot(xp - u(t))$

#4: $\text{perpendic} := yp - n = -\frac{u'(t)}{v'(t)}\cdot(xp - m)$

#5: $(\text{SOLUTIONS}(\text{tangent} \wedge \text{perpendic}, [xp, yp]))_1$

#6: $$\left[\frac{m\cdot u'(t)^2 + v'(t)\cdot u'(t)\cdot(n - v(t)) + u(t)\cdot v'(t)^2}{u'(t)^2 + v'(t)^2}, \frac{v(t)\cdot u'(t)^2 + v'(t)\cdot u'(t)\cdot(m - u(t)) + n\cdot v'(t)^2}{u'(t)^2 + v'(t)^2}\right]$$

It is easy to follow what was done above: we need the tangent and its perpendicular line passing the pedal point (m,n). The intersection point gives a point of the pedal curve (expression #6).

The whole procedure can be compressed into one single function (program):

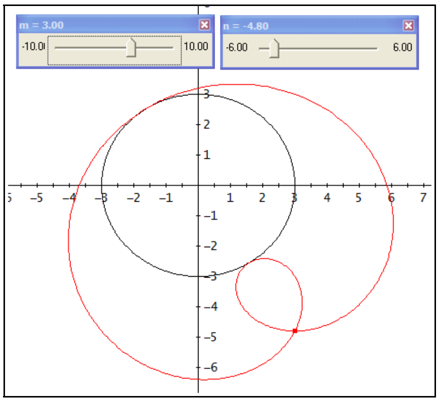
```
ped_c(c, pole, p1, p2, yp, xp, tang, perp) :=
  Prog
    [p1 := pole↓1, p2 := pole↓2]
    c := IF(DIM(c) = 2, c, [t, LIM(c, x, t)], [t, LIM(c, x, t)])
    [u(t) := c↓1, v(t) := c↓2]
    tang := yp - v(t) = v'(t)/u'(t)·(xp - u(t))
    perp := yp - p2 = - u'(t)/v'(t)·(xp - p1)
    (SOLUTIONS(tang ∧ perp, [xp, yp]))↓1
```

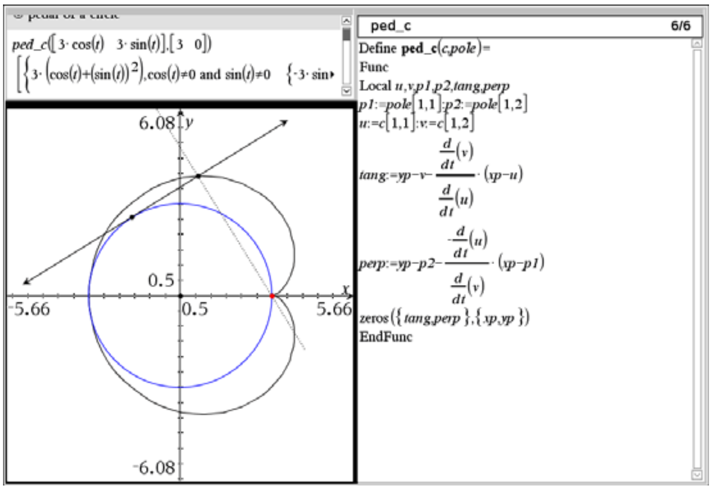
What is the pedal (curve) of a circle?

```
ped_c([3·COS(t), 3·SIN(t)], [m, n])
```

$$[\text{COS}(t)\cdot(3 - n\cdot\text{SIN}(t)) + m\cdot\text{SIN}(t)^2, n\cdot\text{COS}(t)^2 - m\cdot\text{SIN}(t)\cdot\text{COS}(t) + 3\cdot\text{SIN}(t)]$$

We plot the circle, the pedal point and its pedal (curve) after installing sliders for the coordinates of the pedal point.

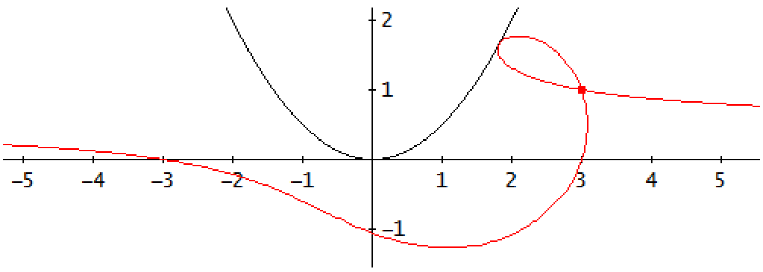




Let's proceed investigating a simple parabola with pedal point = [3,1]:

Same procedure on TI-NspireCAS

$$\text{ped\_c}\left(\frac{x^2}{2}, [3, 1]\right) = \left[\frac{t + 6}{2\cdot(t^2 + 1)} + \frac{t}{2}, \frac{t\cdot(t + 6)}{2\cdot(t^2 + 1)}\right]$$



Which questions can be imagined?

- Tangency point(s)?
- Is there an asymptote?
- Double point(s)?
- Is there an explicit or implicit form of the pedal curve?
- Tangents in the double point?

$$\text{SOLVE}\left(\left[t\_ = \frac{t + 6}{2\cdot(t^2 + 1)} + \frac{t}{2}, \frac{t\_^2}{2} = \frac{t\cdot(t + 6)}{2\cdot(t^2 + 1)}\right], [t\_, t], \text{Real}\right)$$

$$\left[t\_ = 6^{1/3} \wedge t = 6^{1/3}\right]$$

$$\text{SUBST}\left(\left[t\_, \frac{t\_^2}{2}\right], t\_, 6^{1/3}\right) = \left[6^{1/3}, \frac{6^{2/3}}{2}\right]$$

$$\lim_{t\to\infty}\left[\frac{t+6}{2\cdot(t^2+1)}+\frac{t}{2},\ \frac{t\cdot(t+6)}{2\cdot(t^2+1)}\right]=\left[\infty,\ \frac{1}{2}\right]$$

$$y=\boxed{\frac{1}{2}}$$

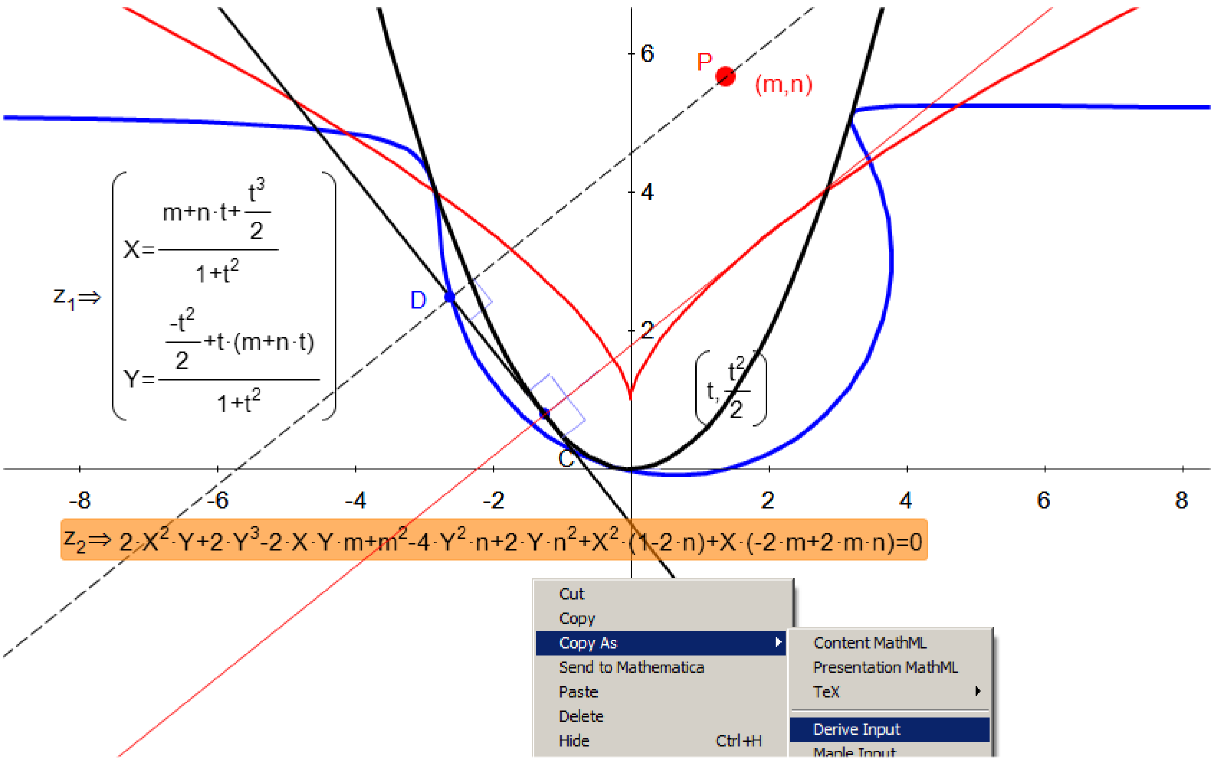
We started with the tangency point and the asymptote. Is there an implicit form possible?

$$\text{SOLVE}\left(\left[x=\frac{t+6}{2\cdot(t^2+1)}+\frac{t}{2},\ y=\frac{t\cdot(t+6)}{2\cdot(t^2+1)}\right],\ [x,\ t],\ \text{Real}\right)$$

$$\left[x=\frac{(y-1)\cdot\sqrt{(-4\cdot y^2+2\cdot y+9)}+3\cdot y}{2\cdot y-1}\wedge t=\frac{\sqrt{(-4\cdot y^2+2\cdot y+9)}-3}{1-2\cdot y},\ x=\frac{(y-1)\cdot\sqrt{(-4\cdot y^2+2\cdot y+9)}-3\cdot y}{1-2\cdot y}\wedge t=\frac{\sqrt{(-4\cdot y^2+2\cdot y+9)}+3}{2\cdot y-1}\right]$$

What we can achieve is the presentation in form of two branches x = …

But there is another possibility, too! We load Geometry Expressions, create the pedal of the parabola and let the software then (with a CAS in its background) calculate the parameter form and implicit form as well:



You see both forms. Expression z_2 can be exported to DERIVE …

$$y\cdot x^{2}\cdot 2 + y^{3}\cdot 2 + m\cdot y\cdot x\cdot(-2) + m^{2} + n\cdot y^{2}\cdot(-4) + n^{2}\cdot y\cdot 2 + (1 +$$
$$n\cdot(-2))\cdot x^{2} + (m\cdot(-2) + n\cdot m\cdot 2)\cdot x = 0$$

$$x^{2}\cdot(2\cdot y - 1) - 6\cdot x\cdot y + 2\cdot y^{3} - 4\cdot y^{2} + 2\cdot y + 9 = 0$$

This expression doesn't look too complicated compared with the two-branch solution from above?. Can we achieve this result with DERIVE, too?

Performing some manipulations with the first expression for *x* (isolating the root, squaring the whole equation, multiplying the equation with its common denominator, …) we finally get:

$$x^{2}\cdot(4\cdot y^{2} - 4\cdot y + 1) + x\cdot(6\cdot y - 12\cdot y^{2}) + 4\cdot y^{4} - 10\cdot y^{3} + 8\cdot y^{2} + 16\cdot y - 9 = 0$$

Dividing this expression by $2y - 1$ gives:

$$\frac{x^{2}\cdot(4\cdot y^{2} - 4\cdot y + 1) + x\cdot(6\cdot y - 12\cdot y^{2}) + 4\cdot y^{4} - 10\cdot y^{3} + 8\cdot y^{2} + 16\cdot y - 9 = 0}{2\cdot y - 1}$$

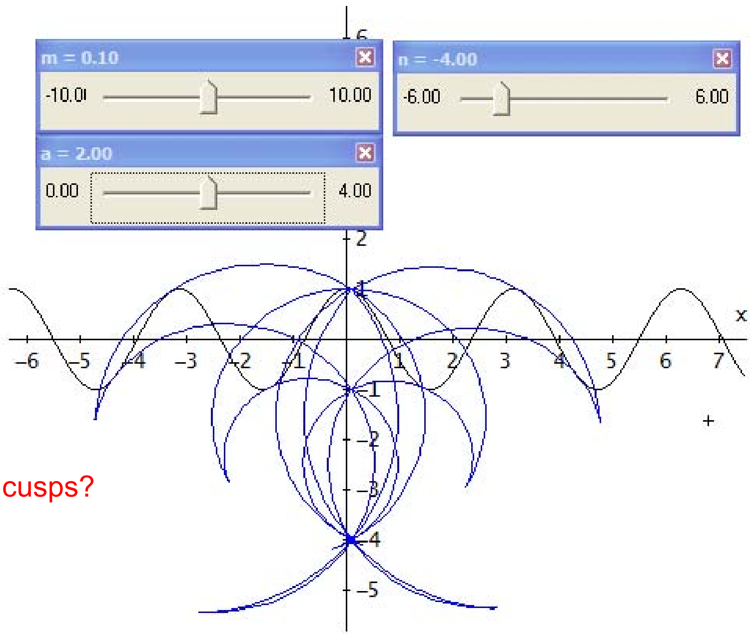
$$x^{2}\cdot(2\cdot y - 1) - 6\cdot x\cdot y + 2\cdot y^{3} - 4\cdot y^{2} + 2\cdot y + 9 = 0$$

Got it !!

The *Geometric Expressions* plot shows additionally the evolute of the parabola. We will need it later.

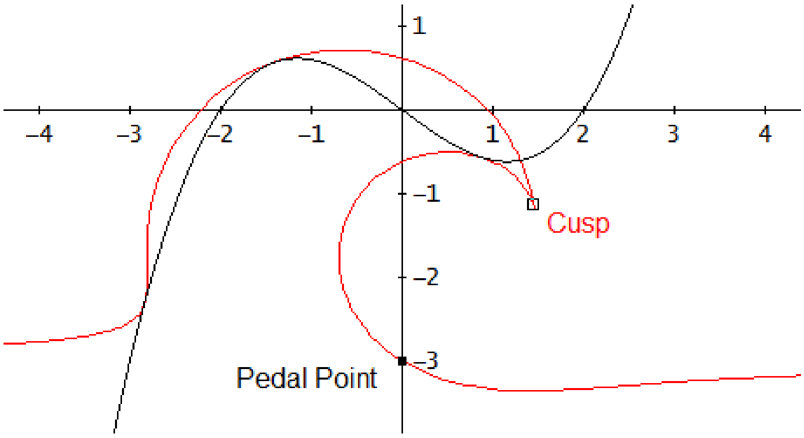
See the family of pedals of a cosine wave (with sliders for all parameters):

ped_c(COS(a·x), [m, n])



What about the cusps?

What regards the cusps, let's have a look at the pedal of a cubic:



The cusp appears when the tangents swings back – this happens in the point of inflection – here at $t = 0$. See the cusp followed by calculation of the tangency points. One of the solution points is an ordinary intersection point,

$$\left[\frac{60}{41}, -\frac{48}{41}\right]$$

[1.463414634, -1.170731707]

$$\text{SOLUTIONS}\left(\left[s = \frac{(3\cdot t^2 - 4)\cdot(2\cdot t^3 - 15)}{9\cdot t^4 - 24\cdot t^2 + 41}, \frac{s^3 - 4\cdot s}{5} = -\frac{27\cdot t^4 + 10\cdot t^3 - 72\cdot t^2 + 48}{9\cdot t^4 - 24\cdot t^2 + 41}\right], [s, t], \text{Real}\right)$$

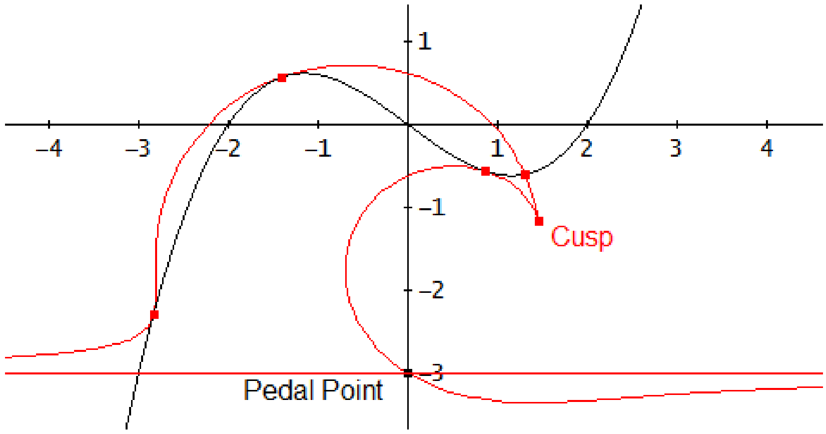
$$\begin{bmatrix} -1.411 & -1.411 \\ 1.306 & -0.6530 \\ 0.8624 & 0.8624 \\ -2.836 & -2.836 \end{bmatrix}$$

$$\text{VECTOR}\left(\left[s, \frac{s^3 - 4\cdot s}{5}\right], s, [-1.411, 1.306, 0.8624, -2.836]\right)$$

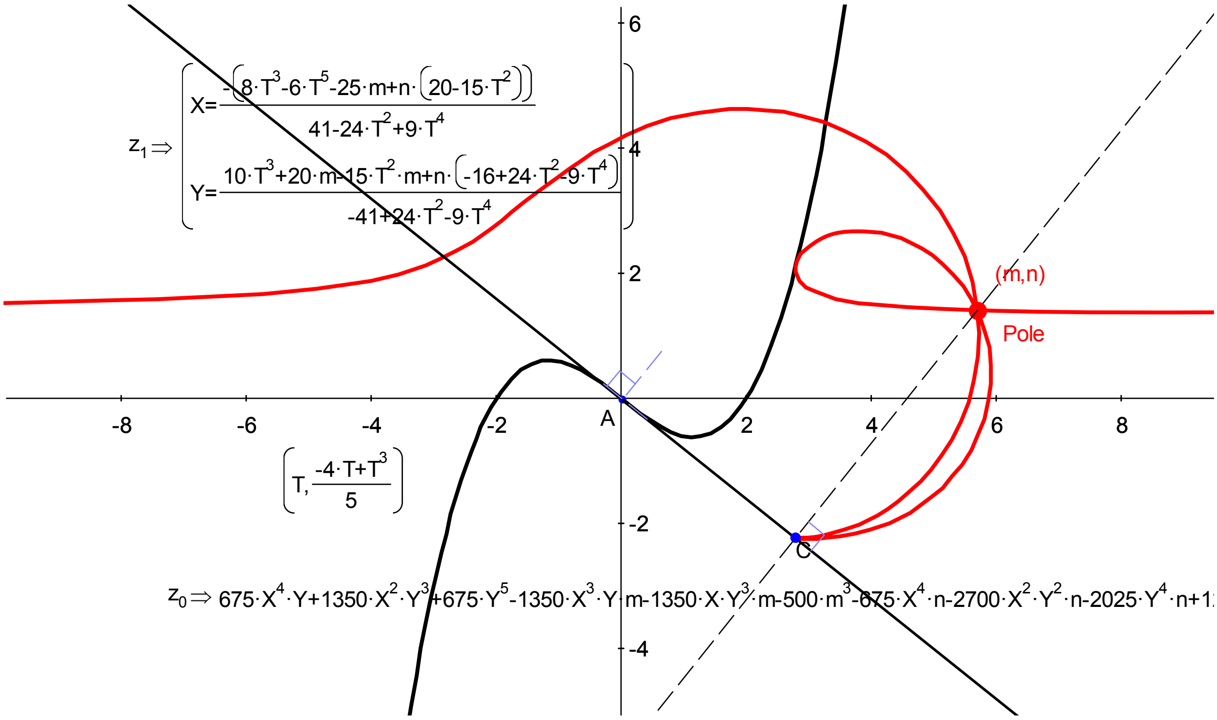
$$\begin{bmatrix} -1.411 & 0.5669 \\ 1.306 & -0.5992 \\ 0.8624 & -0.5616 \\ -2.836 & -2.293 \end{bmatrix}$$

Finally, the asymptote of the pedal is $y = -3$.

$$\lim_{t\to\infty}\left[\frac{(3\cdot t^2-4)\cdot(2\cdot t^3-15)}{9\cdot t^4-24\cdot t^2+41}, -\frac{27\cdot t^4+10\cdot t^3-72\cdot t^2+48}{9\cdot t^4-24\cdot t^2+41}\right] = [\infty, -3]$$



Geometry Expressions provides the implicit form of this interesting pedal curve:



Here we have only one tangency point, but the pedal point turns out to be a triple point.

Open question: Which position of the point on given curve C results in passing the pedal point on the pedal curve?

At the end of this paragraph we would like to find the tangents in the triple point. The graph demonstrates that the pedal point (Pole) is the triple point. We will assume that its coordinates are (5,1). Then we have to find the parameter values which make the second coordinate of the pedal curve equal 1:

$$pc := ped\_c\left(\left[t, \frac{t^3 - 4\cdot t}{5}\right], [5, 1]\right)$$

$$pc := \left[\frac{24\cdot t^3 + 45\cdot t^2 - 82\cdot t + 315}{3\cdot(9\cdot t^4 - 24\cdot t^2 + 41)} + \frac{2\cdot t}{3}, \frac{9\cdot t^4 - 10\cdot t^3 + 51\cdot t^2 - 84}{9\cdot t^4 - 24\cdot t^2 + 41}\right]$$

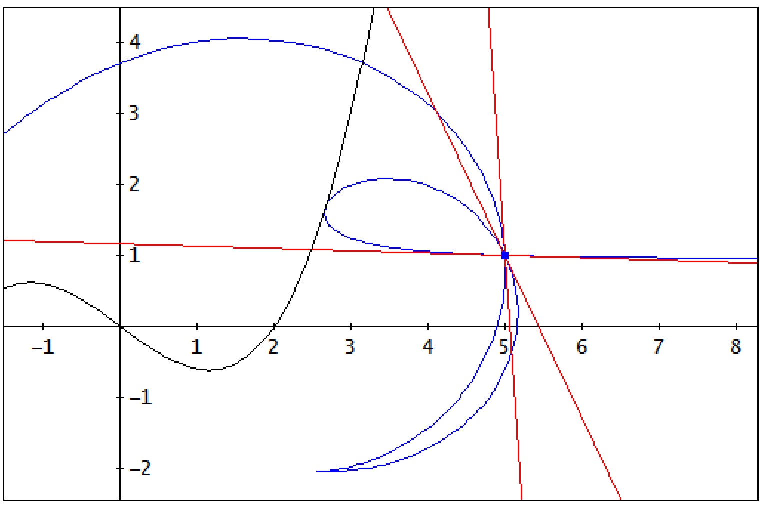
$$\mathrm{SOLVE}(pc_2 = 1, t)$$

$$t = -1.19874 \lor t = 1.43570 \lor t = 7.26304$$

With the slope defined and evaluated for the respective parameter values it is easy work to find the three tangents.

$$slp := \frac{\frac{d}{dt} pc_2}{\frac{d}{dt} pc_1}$$

```
VECTOR(slp, t, [-1.199, 1.436, 7.263])

[-15.8912, -2.28431, -0.0324145]

VECTOR(y = m·(x - 5) + 1, m, [-15.8912, -2.28431, -0.0324145])

EXPAND(VECTOR(y = m·(x - 5) + 1, m, [-15.8912, -2.28431, -0.0324145]))

[y = 80.456 - 15.8912·x, y = 12.4215 - 2.28431·x, y = 1.16207 - 0.0324145·x]
```



The cubic (black), its pedal curve (blue) and the three tangents in the pedal point (red).

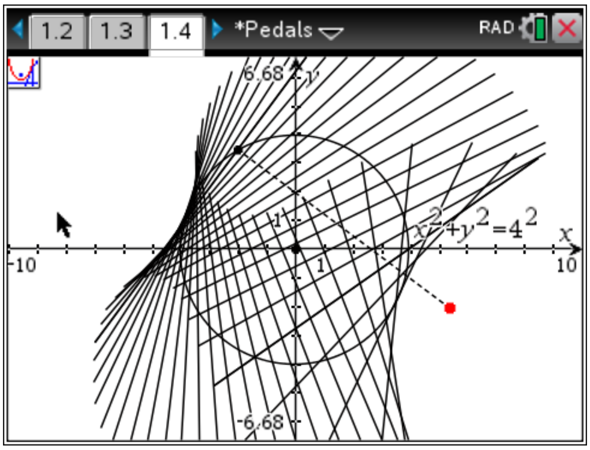
2 What is a Negative Pedal Curve?

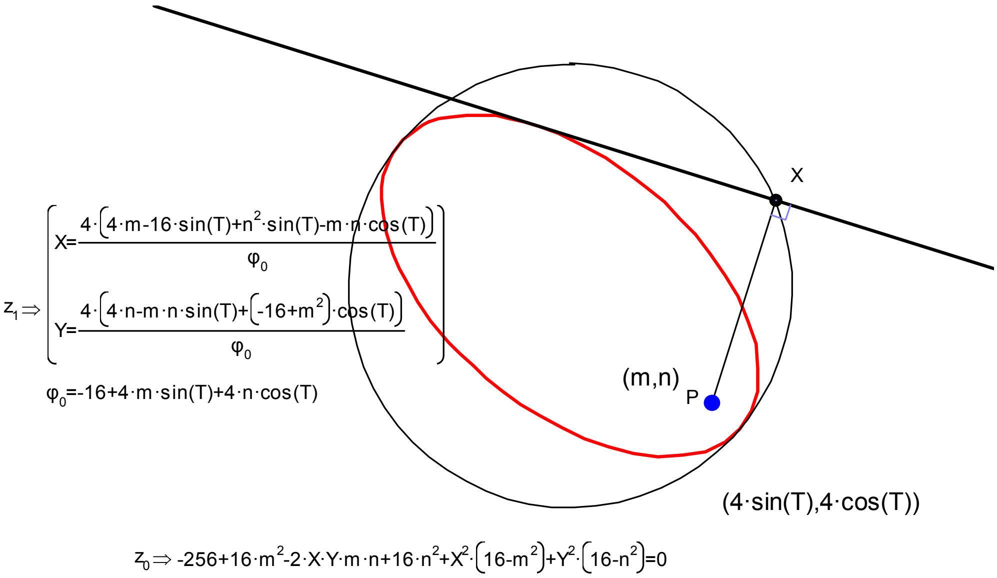
For a plane curve *C* and a given fixed point *P* (the pedal point) the **negative pedal curve of *C*** is the envelope of all lines which are perpendicular to lines *PX* with points X being points on *C*.

We will demonstrate this using TI-NspireCAS first: What is the negative pedal of a circle?

Unfortunately the locus tool of TI-Nspire does not produce the envelope but only the family of the perpendicular lines – which gives the impression that the envelope is a hyperbola.

So let's try another tool before doing the calculation work, again Geometry Expressions.





The pedal point inside the circle gives an ellipse. The implicit form of the negative pedal shows that it must be a conic.

We will do the calculation now:

Reminder:
Finding the envelope of a family of curves *F*(x(*t*), *y*(*t*)) depending on the parameter *t*:

$$\left.\begin{aligned} &F(x(t),y(t))=0 \\ &\frac{dF(x)(t),y(t)}{dt}=0 \end{aligned}\right\}$$

Solve for *x* and *y* or eliminate *t* for the implicit form.

```
nped_c(c, pole, p1, p2, ray, x, y) :=
  Prog
    [p1 := pole↓1, p2 := pole↓2]
    c := IF(DIM(c) = 2, c, [t, LIM(c, x, t)], [t, LIM(c, x, t)])
    [u(t) := c↓1, v(t) := c↓2]
    ray := - (u(t) - p1)/(v(t) - p2)·(x - u(t)) - y + v(t)
    RETURN (SOLUTIONS(ray = 0 ∧ ∂(ray, t) = 0, [x, y]))↓1
```

The negative pedal of a circle with a point inside its circumference results – after some not too difficult manipulations – in an ellipse.

nped_c([4·COS(t), 4·SIN(t)], [3, 0])

$$\left[\frac{4\cdot(4\cdot\mathrm{COS}(t)-3)}{4-3\cdot\mathrm{COS}(t)}, \frac{7\cdot\mathrm{SIN}(t)}{4-3\cdot\mathrm{COS}(t)}\right]$$

$$16\cdot y^2 = 7\cdot(16 - x^2)$$

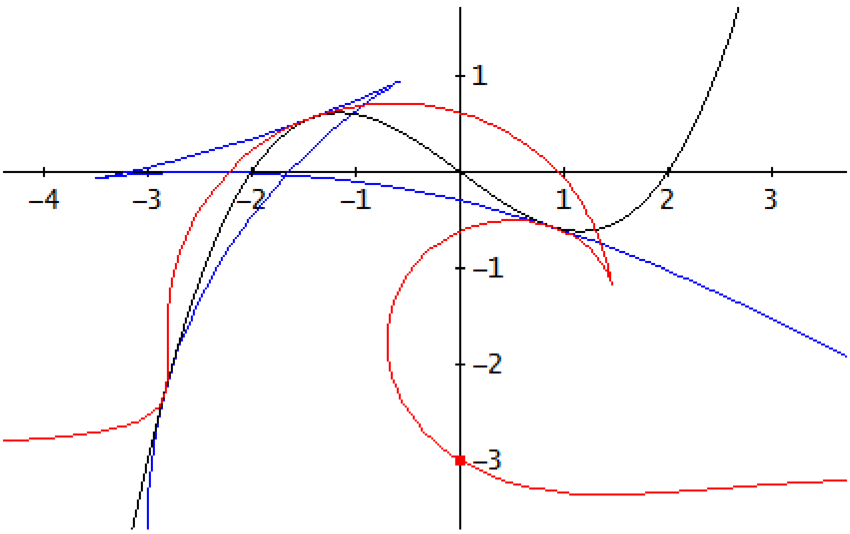
The negative pedal of a pedal leads back to the original curve which is demonstrated below:

$$\mathrm{nped\_c}\left(\mathrm{ped\_c}\left(\frac{x^3 - 4\cdot x}{5}, [0, -3]\right), [0, -3]\right)$$

$$\left[t, \frac{t\cdot(t^2-4)}{5}\right]$$

We plot the cubic, its pedal and its negative pedal:

$$\mathrm{nped\_c}\left(\frac{x^3 - 4\cdot x}{5}, [0, -3]\right)$$



Questions: The tangency points with the original curve seem to be the same?

Now we have two cusps. What is their position?

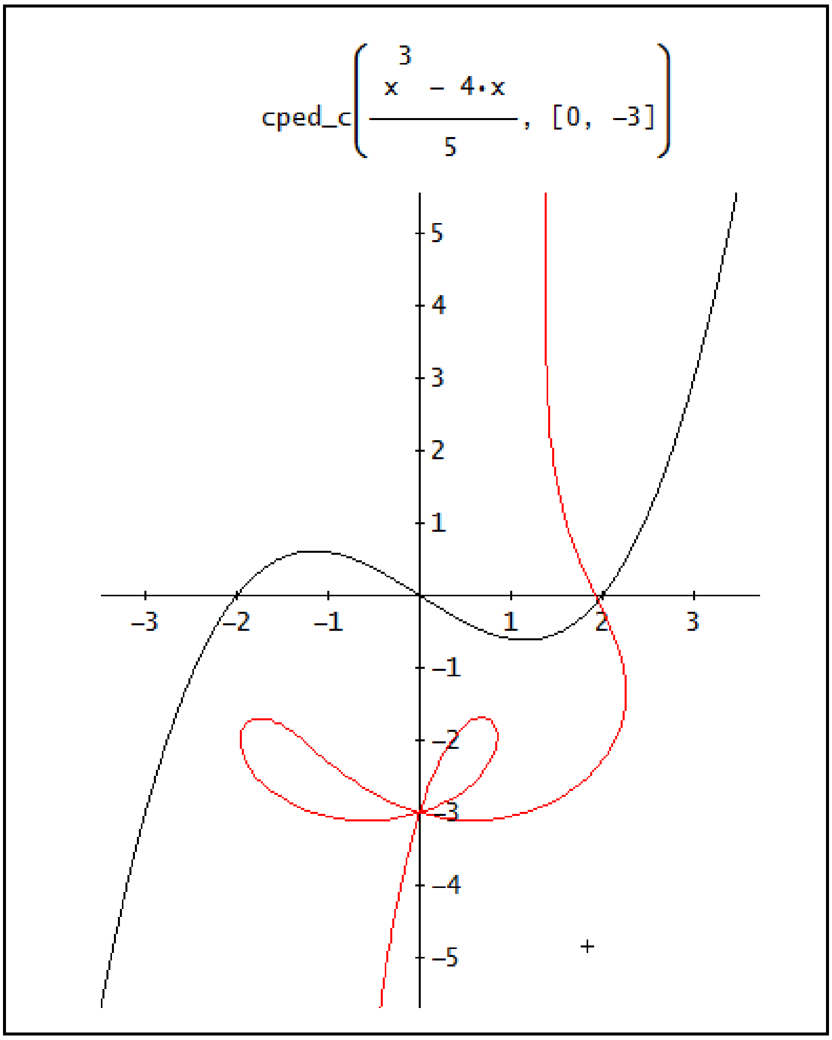
3 What is a Contra Pedal Curve?

We take the definition of the pedal curve from above and replace *tangent* by ***normal***:

For a plane curve *C* and a given fixed point *P* the **contra pedal curve of *C*** is the locus of all points *X* with line *PX* being perpendicular to the normal *n* in *X*, point of curve *C*.

```
cped_c(c, pole, p1, p2, xp, yp, norm, perpnorm, cpc) :=
  Prog
    [p1 := pole↓1, p2 := pole↓2]
    c := IF(DIM(c) = 2, c, [t, LIM(c, x, t)], [t, LIM(c, x, t)])
    [u(t) := c↓1, v(t) := c↓2]
    norm := yp - v(t) = u'(t)/v'(t)·(u(t) - xp)
    perpnorm := yp - p2 = v'(t)/u'(t)·(xp - p1)
    cpc := (SOLUTIONS(perpnorm ∧ norm, [xp, yp]))↓1
```

The contra pedal of our cubic from above:



The Contra Pedal of a curve *C* is the Pedal of the Evolute of the curve *C*!

```
evolute(c, xp, yp, norm, dnorm, ev) :=
  Prog
    c := IF(DIM(c) = 2, c, [t, LIM(c, x, t)], [t, LIM(c, x, t)])
    [u(t) := c↓1, v(t) := c↓2]
    norm := yp - v(t) - u'(t)/v'(t)·(u(t) - xp) = 0
    dnorm := ∂(norm, t)
    ev := (SOLUTIONS(norm ∧ dnorm, [xp, yp]))↓1
```

$$\mathrm{evolute}\left(\frac{x^3 - 4\cdot x}{5}\right) = \left[-\frac{27\cdot t^6 - 108\cdot t^4 + 69\cdot t^2 - 164}{150\cdot t}, \frac{15\cdot t^4 - 48\cdot t^2 + 41}{30\cdot t}\right]$$

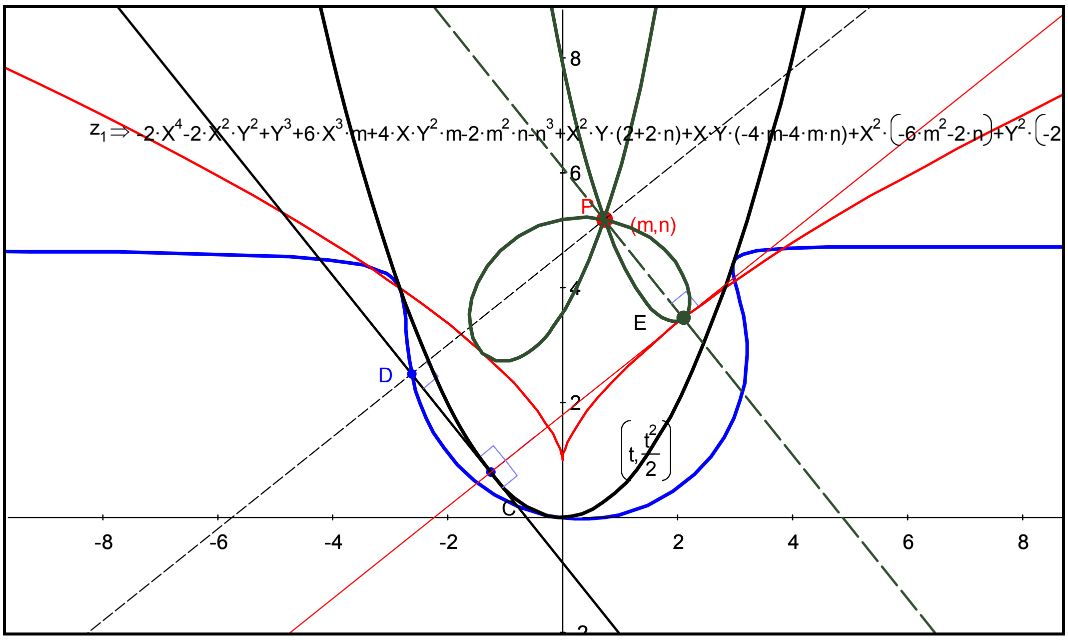
$$\mathrm{ped\_c}\left(\left[-\frac{27\cdot t^6 - 108\cdot t^4 + 69\cdot t^2 - 164}{150\cdot t}, \frac{15\cdot t^4 - 48\cdot t^2 + 41}{30\cdot t}\right], [0, -3]\right)$$

$$\left[\frac{t}{3} - \frac{24\cdot t^3 - 135\cdot t^2 - 82\cdot t + 180}{3\cdot(9\cdot t^4 - 24\cdot t^2 + 41)}, \frac{9\cdot t^7 - 60\cdot t^5 + 187\cdot t^3 - 164\cdot t - 375}{5\cdot(9\cdot t^4 - 24\cdot t^2 + 41)}\right]$$

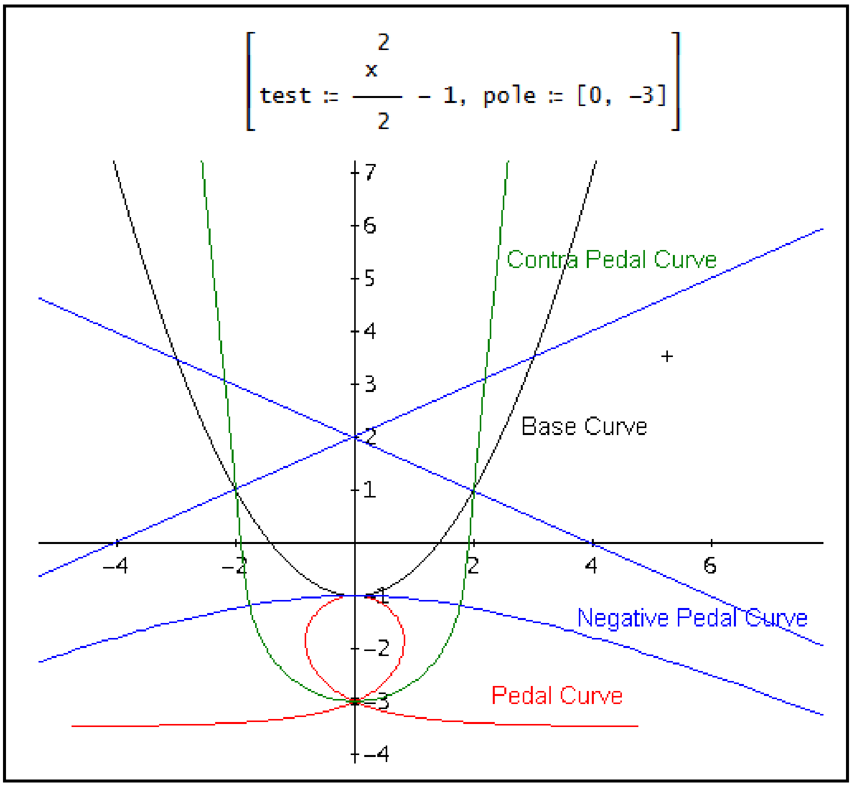
$$\mathrm{cped\_c}\left(\frac{x^3 - 4\cdot x}{5}, [0, -3]\right)$$

$$\left[\frac{t}{3} - \frac{24\cdot t^3 - 135\cdot t^2 - 82\cdot t + 180}{3\cdot(9\cdot t^4 - 24\cdot t^2 + 41)}, \frac{9\cdot t^7 - 60\cdot t^5 + 187\cdot t^3 - 164\cdot t - 375}{5\cdot(9\cdot t^4 - 24\cdot t^2 + 41)}\right]$$

The red curve is the evolute of the parabola and the green one is its contra pedal.



The plot shows pedal, negative pedal and contra pedal of a parabola united:

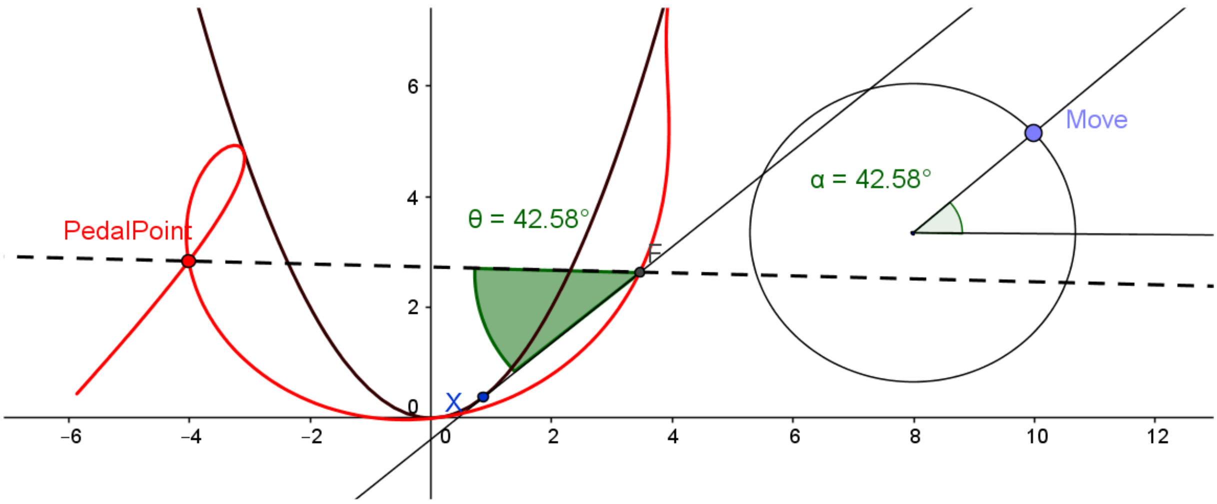


4 What is a Skew Pedal Curve?

We now skew the discussion slightly by examining the locus of the intersection of the tangent to the parabola with the line through P at angle θ to the tangent.

Phil Todd
Geometry Expressions

The following screen shot shows a GeoGebra presentation of a skew pedal curve.

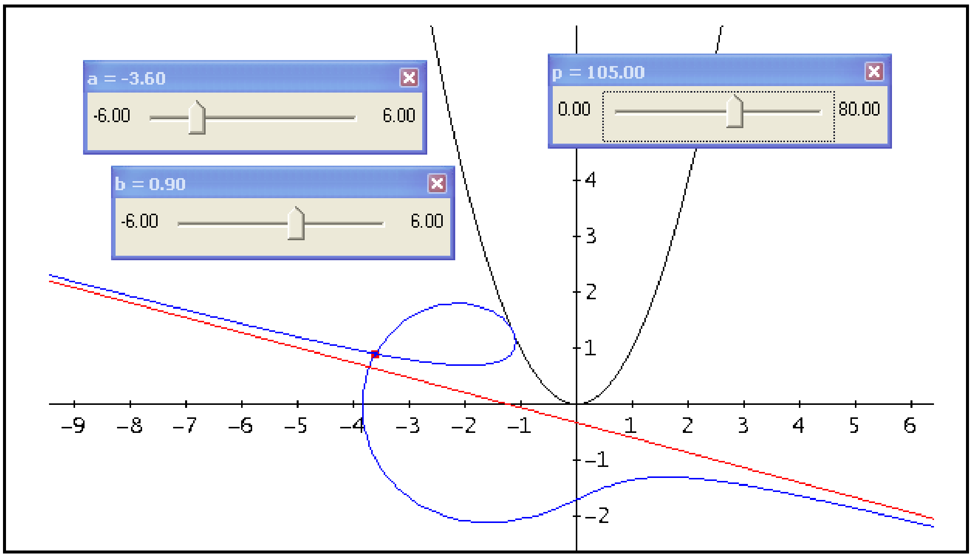


Angle θ can be changed by moving point Move on the circumference of the circle.

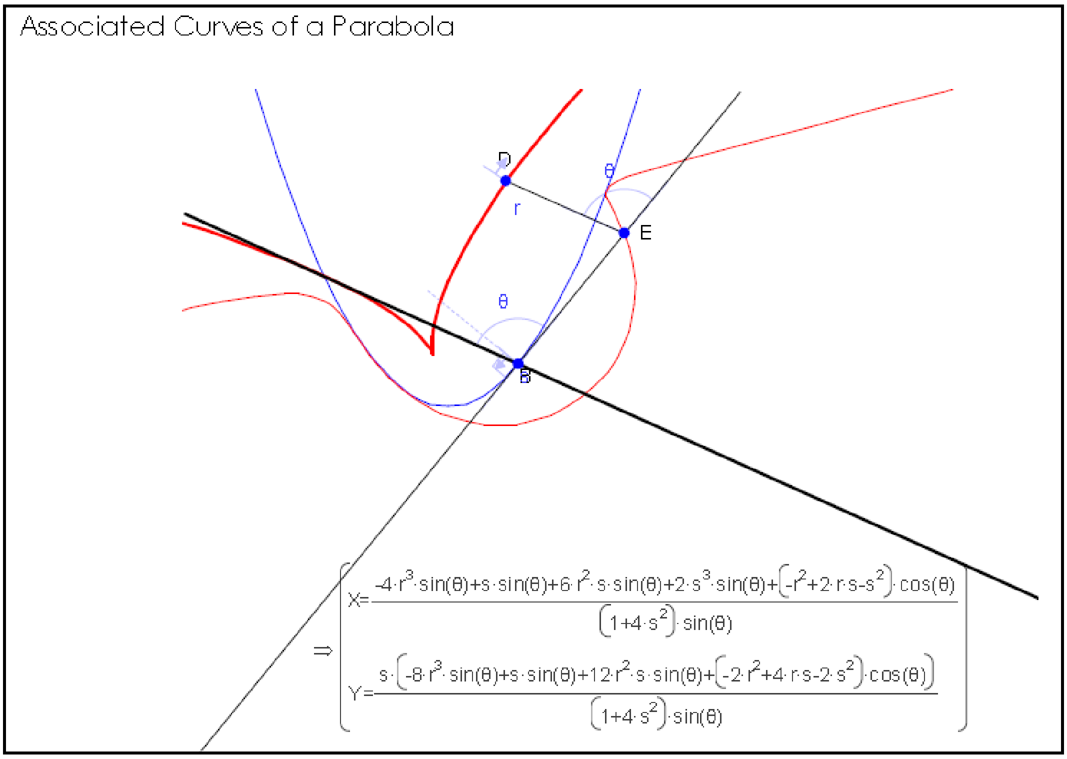
Sliders for the coordinates (*a*,*b*) of the pedal point and *p* for the angle (in °) enable studying the variations of the skew pedal of the parabola (or of any other curve, of course).

$$\text{skew\_pedal\_curve}(x^2, [a, b], p°)$$

$$\left[\frac{a + 2\cdot t\cdot(b + t^2)}{4\cdot t^2 + 1} - \frac{(2\cdot a\cdot t - b - t^2)\cdot \mathrm{COT}(p)}{4\cdot t^2 + 1}, \frac{t\cdot(2\cdot a + t\cdot(4\cdot b - 1))}{4\cdot t^2 + 1} - \frac{2\cdot t\cdot(2\cdot a\cdot t - b - t^2)\cdot \mathrm{COT}(p)}{4\cdot t^2 + 1}\right]$$



Finally we will show the *Geometry Expressions* product presenting the parameter form of the skew pedal curve (pedal point (*r*,*s*) and angle θ.



5 The Pedal of the Pedal of the Pedal …

We make use of the wonderful and versatile ITERATES-command of DERIVE.

Instead of

$$\#65: \quad \mathrm{ped\_c}\left(\frac{x^2}{2} - 1, [0, -3]\right)$$

$$\#66: \quad \left[\frac{t\cdot(t^2 - 4)}{2\cdot(t^2 + 1)}, -\frac{7\cdot t^2 + 2}{2\cdot(t^2 + 1)}\right]$$

$$\#67: \quad \mathrm{ped\_c}\left(\left[\frac{t\cdot(t^2 - 4)}{2\cdot(t^2 + 1)}, -\frac{7\cdot t^2 + 2}{2\cdot(t^2 + 1)}\right], \mathrm{pole}\right)$$

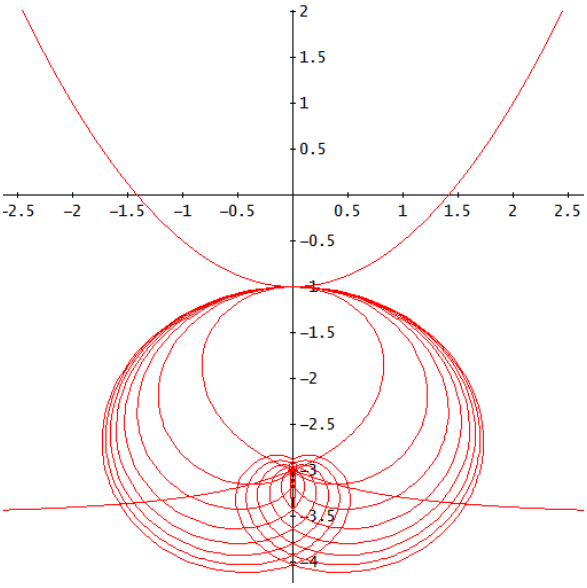
$$\#68: \quad \left[-\frac{5\cdot t\cdot(t^2 - 4)}{(t^2 + 1)^2\cdot(t^4 + 12\cdot t^2 + 16)}, -\frac{7\cdot t^8 + 83\cdot t^6 + 202\cdot t^4 + 408\cdot t^2 + 32}{2\cdot(t^2 + 1)^2\cdot(t^4 + 12\cdot t^2 + 16)}\right]$$

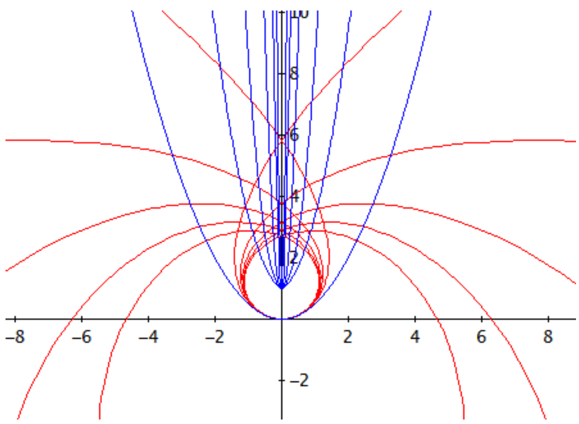
we can enter

$$\mathrm{ITERATES}\left(\mathrm{ped\_c}(w, [0, -3]), w, \frac{x^2}{2} - 1, 8\right)$$

$$\mathrm{ITERATES}\left(\mathrm{nped\_c}(w, [0, 1]), w, \frac{x^2}{2}, 5\right)$$

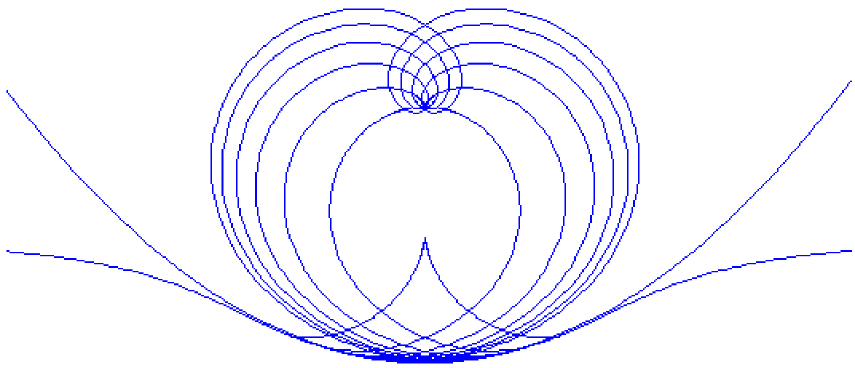
$$\mathrm{ITERATES}\left(\mathrm{cped\_c}(w, [0, 1]), w, \frac{x^2}{2}, 5\right)$$

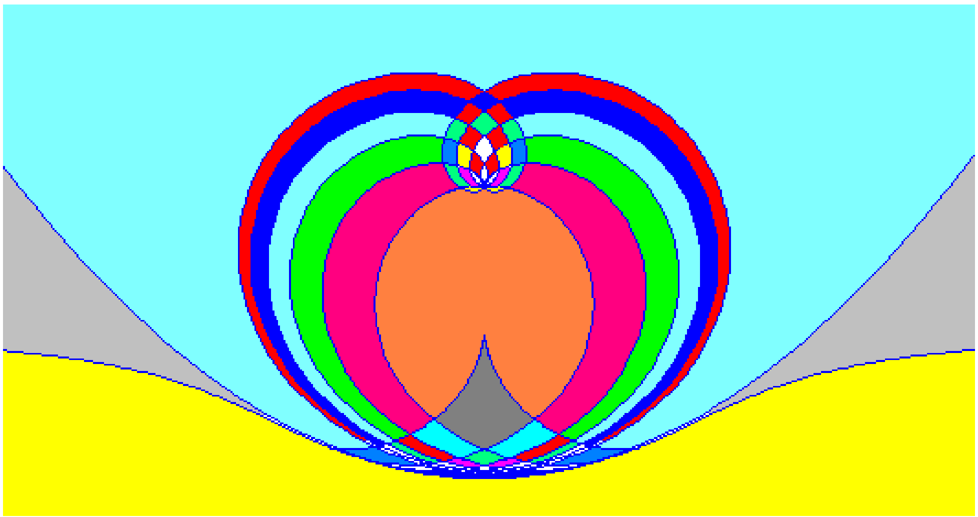




in order to receive the first eight iterates of the pedal (left) and the first five iterates of the negative pedal and the contra pedal (right).

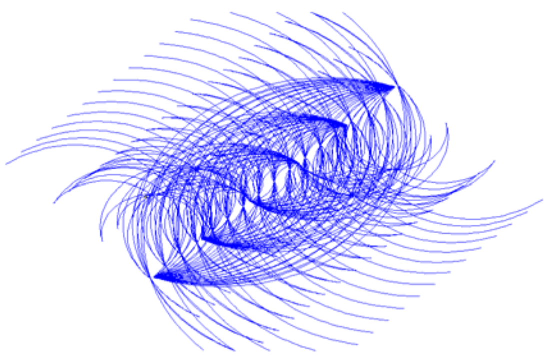
Students might be inspired to make their own experiments using well know base curves. DERIVE or graphic programs support creating nice pictures:

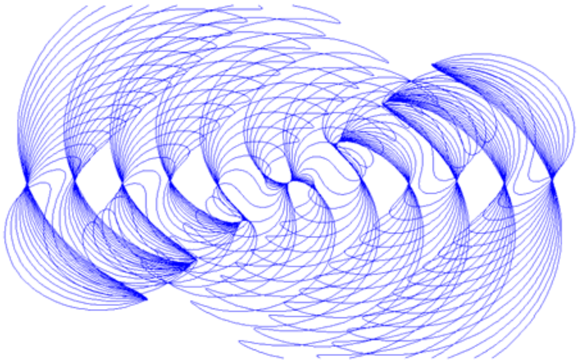




Families of pedals and contra pedals create great patterns. See two examples below:

```
VECTOR(ped_c(SIN(x), [n, n]), n, -8, 8, 1)

VECTOR(cped_c(SIN(x), [n, n]), n, -8, 8, 1)
```



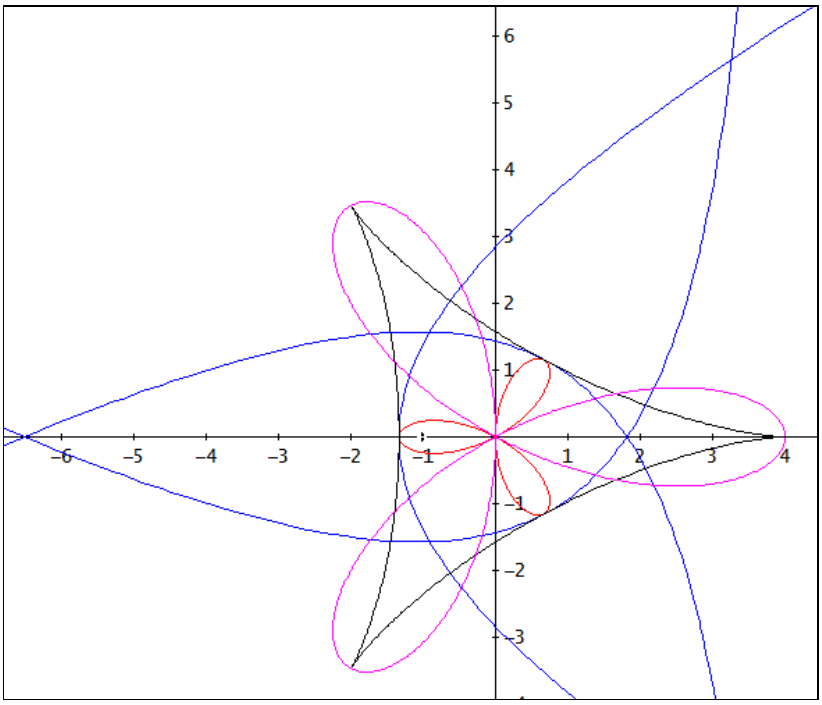


6 Relationships

Curve	Pedal Point	Pedal Curve
Circle	On circumference	Cardioid
Cardiod	Cusp	Cayley`s Sextic
Circle	Other point (not centre)	Limacon
Astroid	Center	Quadrifolium
Ellipse, Hyperbola	Focus	Circle
Parabola	Focus	Line
Parabola	Vertex	Cissoid
Cissoid	Focus	Cardioid
Epicycloid	Centre	Rose

This is only a short selection. Large lists can be found in the internet.

Cycloid (black) with pedal curve, negative pedal curve and contra pedal curve. Pedal point is in the origin.



7 Pedal Curves in Space

The curve which is the locus of the intersection points of the perpendiculars let fall from a fixed point *P* upon the tangents to a given curve *C*.

We will produce the pedal curve of an elliptical helix step by step:

Space Curve with parameter φ

#1: $hel := [3\cdot COS(\phi), 2\cdot SIN(\phi), 0.3\cdot\phi]$

Tangent with parameter t

#2: $tang(c) := c + t\cdot\frac{d}{d\phi} c$

#3: $tang(hel)$

#4: $\left[3\cdot COS(\phi) - 3\cdot t\cdot SIN(\phi), 2\cdot t\cdot COS(\phi) + 2\cdot SIN(\phi), \frac{3\cdot t}{10} + \frac{3\cdot\phi}{10}\right]$

Plane perpendicular to the tangent passing the pedal point

#5: $perp\_pl(c, ppt) := [x, y, z]\cdot\frac{d}{d\phi} c = ppt\cdot\frac{d}{d\phi} c$

#6: $perp\_pl(hel, [4, 4, 4])$

#7: $2\cdot y\cdot COS(\phi) - 3\cdot x\cdot SIN(\phi) + \frac{3\cdot z}{10} = 8\cdot COS(\phi) - 12\cdot SIN(\phi) + \frac{6}{5}$

Intersection tangent and plane giving the t-value
for the pedal point on the tangent

#8: $param(c, ppt) := (SOLUTIONS(SUBST(perp\_pl(c, ppt), [x, y, z], tang(c)), t))_1$

#9: $param(hel, [4, 4, 4])$

#10: $\frac{100\cdot COS(\phi)\cdot(5\cdot SIN(\phi) + 8) - 3\cdot(400\cdot SIN(\phi) + 3\cdot\phi - 40)}{500\cdot SIN(\phi)^2 + 409}$

Fixing the point on the tangent (as function of φ) –
– results in the pedal curve

#11: $ped\_c(c, ppt) := SUBST(tang(c), t, param(c, ppt))$

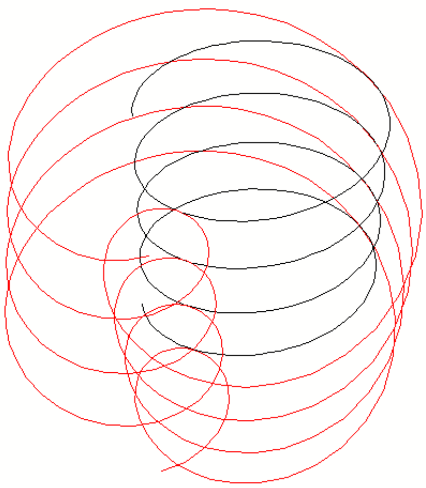
#12: $ped\_c(hel, [4, 4, 4])$

#13: $$\left[\frac{3\cdot(3\cdot SIN(\phi)\cdot(400\cdot SIN(\phi) + 3\cdot\phi - 40) - COS(\phi)\cdot(800\cdot SIN(\phi) - 409))}{500\cdot SIN(\phi)^2 + 409}, \frac{2\cdot(800\cdot COS(\phi)^2 - 3\cdot COS(\phi)\cdot(400\cdot SIN(\phi) + 3\cdot\phi - 40) + 909\cdot SIN(\phi))}{500\cdot SIN(\phi)^2 + 409}, \frac{6\cdot(5\cdot COS(\phi)\cdot(5\cdot SIN(\phi) + 8) + 25\cdot\phi\cdot SIN(\phi)^2 - 60\cdot SIN(\phi) + 2\cdot(10\cdot\phi + 3))}{500\cdot SIN(\phi)^2 + 409}\right]$$

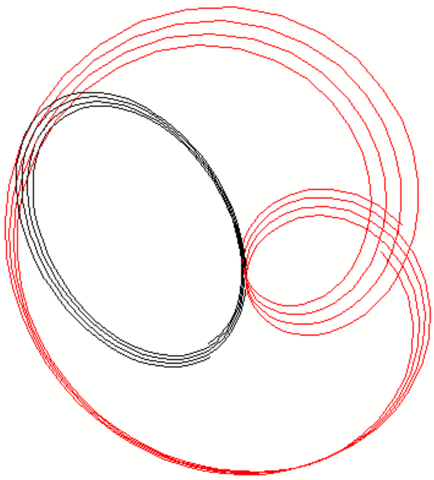
The respective function and some applications:

```
spped_c(c, par, ppt, tang, perp_pl, τ) :=
  Prog
    tang := c + t_·∂(c, par)
    perp_pl := [x, y, z]·∂(c, par) = ppt·∂(c, par)
    τ := (SOLUTIONS(SUBST(perp_pl, [x, y, z], tang), t_))↓1
    SUBST(tang, t_, τ)
```

```
spped_c(hel, φ, [4, 4, 4])
```



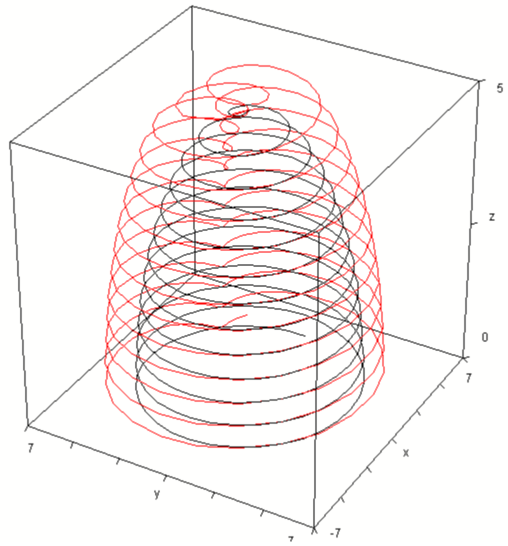
Another view of the curve:



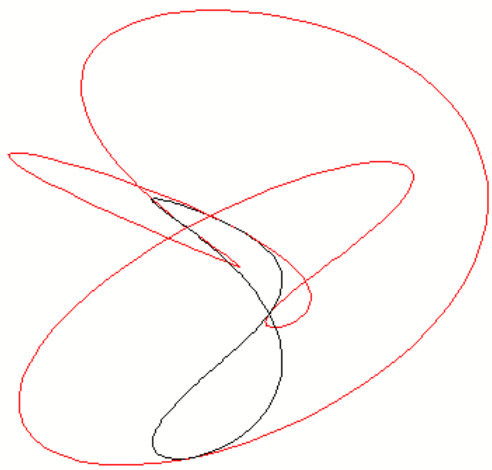
A spherical Snail

$$sphs := \left[\sqrt{(25 - \phi^2)}\cdot COS(5\cdot\pi\cdot\phi),\ \sqrt{(25 - \phi^2)}\cdot SIN(5\cdot\pi\cdot\phi),\ \phi\right]$$

```
spped_c(sphs, φ, [3, 3, 3])
```



Window of Vivian & its pedal curve



8 Pedal Surface

The surface curve which is the locus of the intersection points of the perpendiculars let fall from a fixed point *P* upon the tangent planes of a given surface.

The procedure is not too difficult. So we squeeze it into a small program – or function on TI-NspireCAS:

We fill find the pedal surface of an ellipsoid (variable lengths of axes) with respect to a variable pedal point – great job for CAS and sliders.

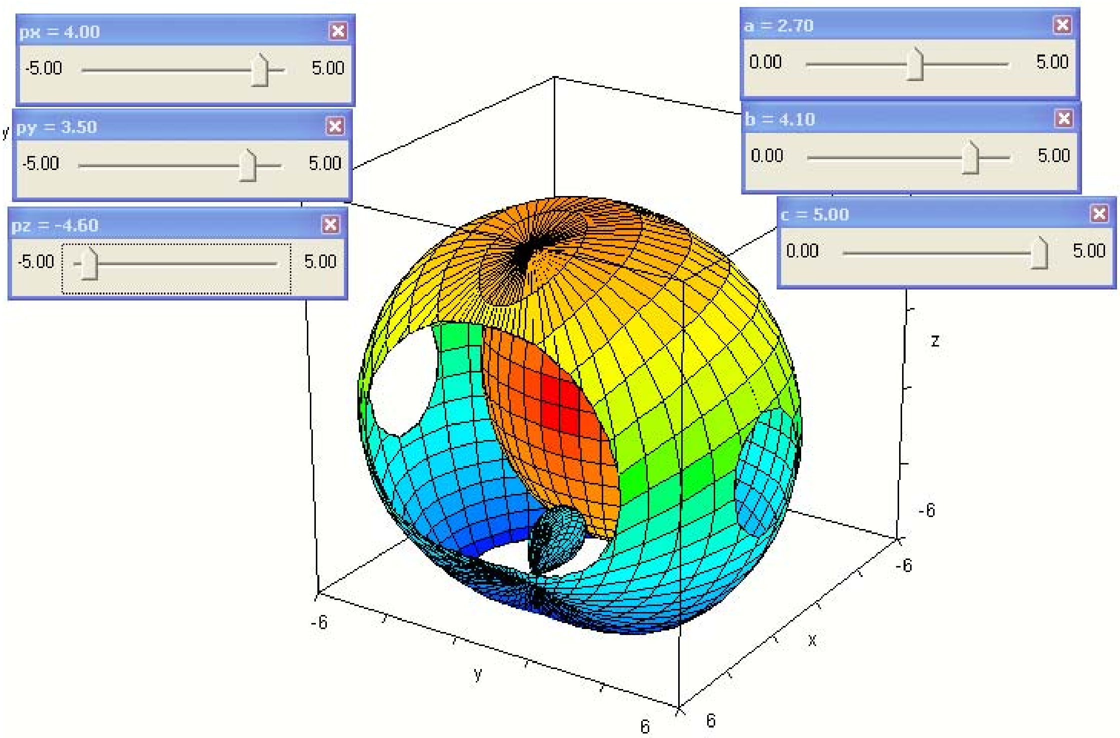
```
pedsurf(surf, par1, par2, ppt, tpl, perp, t_) :=
  Prog
    tpl := surf + α ∂(surf, par1) + β ∂(surf, par2)
    perp := ppt + γ ∂(surf, par1) × ∂(surf, par2)
    t_ := (SOLUTIONS(tpl = perp, [α, β, γ]))↓1↓3
    SUBST(perp, γ, t_)

ell := [a·COS(u)·COS(v), b·COS(u)·SIN(v), c·SIN(u)]

pedsurf(ell, u, v, [px, py, pz])
```

$$\left[-\frac{9\cos(u)^2\sin(v)(2\,py\cos(v)-px\sin(v))+12\cos(u)\cos(v)(pz\sin(u)-3)-4\,px\sin(u)^2}{\cos(u)^2(27\cos(v)^2+5)+4},\right.$$

$$\frac{2(9\cos(u)^2\cos(v)(2\,py\cos(v)-px\sin(v))+3\cos(u)\sin(v)(3-pz\sin(u))+2\,py\sin(u)^2)}{\cos(u)^2(27\cos(v)^2+5)+4},$$

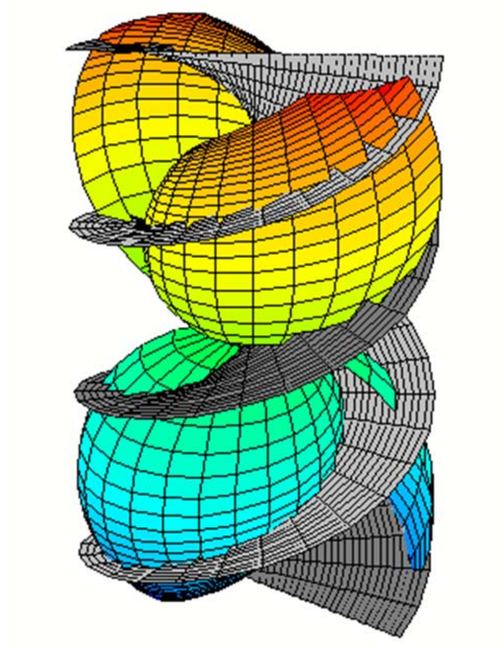
$$\left.\frac{3(3\,pz\cos(u)^2(3\cos(v)^2+1)-2\sin(u)\cos(u)(2\,px\cos(v)+py\sin(v))+4\sin(u))}{\cos(u)^2(27\cos(v)^2+5)+4}\right]$$

Sliders, sliders, sliders, …



Helicoid

```
[v·COS(u), v·SIN(u), u]

pedsurf([v·COS(u), v·SIN(u), u], u, v, [0, 0, 0])
```



Question:

How does this surface change with the pedal point gliding along the *z*-axis?

(An exercise for TI-NspireCAS!)

$xp3(t,u):=t\cdot\cos(u)$ *Done*

$yp3(t,u):=t\cdot\sin(u)$ *Done*

$zp3(t,u):=u$ *Done*

$comp2:=ped\_surf([v\cdot\cos(u)\quad v\cdot\sin(u)\quad u],u,v,[0\quad 0\quad 0])$

$$\left[\frac{t\cdot\sin(t)\cdot u}{u^2+1}\quad \frac{-t\cdot\cos(t)\cdot u}{u^2+1}\quad \frac{t\cdot u^2}{u^2+1}\right]$$

$xp4(t,u):=comp2[1,1]$ *Done*

$yp4(t,u):=comp2[1,2]$ *Done*

$zp4(t,u):=comp2[1,3]$ *Done*

$comp3:=ped\_surf([v\cdot\cos(u)\quad v\cdot\sin(u)\quad u],u,v,[0\quad 0\quad pz])$

$$\left[\frac{(t-pz)\cdot\sin(t)\cdot u}{u^2+1}\quad \frac{-(t-pz)\cdot\cos(t)\cdot u}{u^2+1}\quad \frac{t\cdot u^2+pz}{u^2+1}\right]$$

$xp5(t,u):=\frac{(t-pz)\cdot\sin(t)\cdot u}{u^2+1}$ *Done*

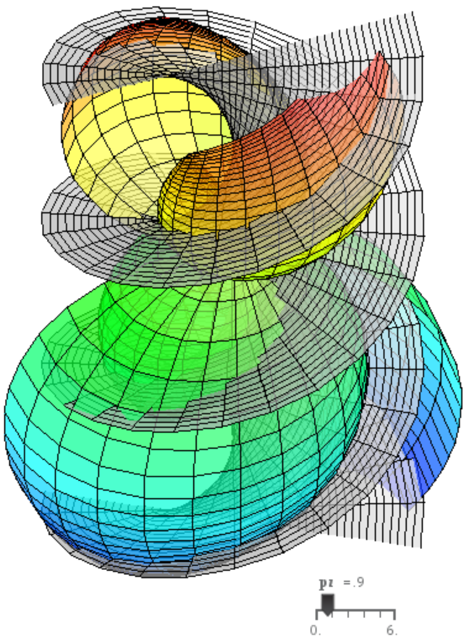
$yp5(t,u):=\frac{-(t-pz)\cdot\cos(t)\cdot u}{u^2+1}$ *Done*

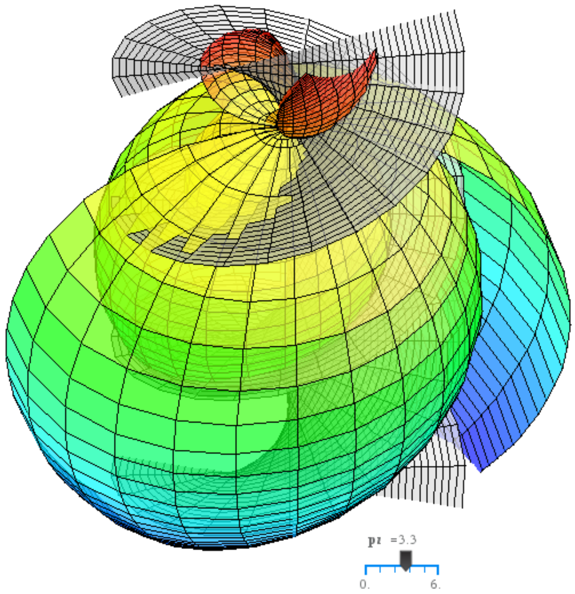
$zp5(t,u):=\frac{t\cdot u^2+pz}{u^2+1}$ *Done*

ped_surf 0/9

Define **ped_surf**$(surf,v1,v2,pole)=$
Func
Local $tpl,perp,d\_v1,d\_v2,eq1,eq2,eq3,sol$
$tpl:=surf+\alpha\cdot\frac{d}{dv1}(surf)+\beta\cdot\frac{d}{dv2}(surf)$
$perp:=pole+\gamma\cdot \text{crossP}\left(\frac{d}{dv1}(surf),\frac{d}{dv2}(surf)\right)$
$eq1:=tpl[1,1]-perp[1,1]$
$eq2:=tpl[1,2]-perp[1,2]$
$eq3:=tpl[1,3]-perp[1,3]$
$sol:=\text{zeros}(\{eq1,eq2,eq3\},\{\alpha,\beta,\gamma\})[1\quad 3]$
$surf:=pole+sol\cdot \text{crossP}\left(\frac{d}{dv1}(surf),\frac{d}{dv2}(surf)\right)|v1=t$
$surf|v2=u$
EndFunc

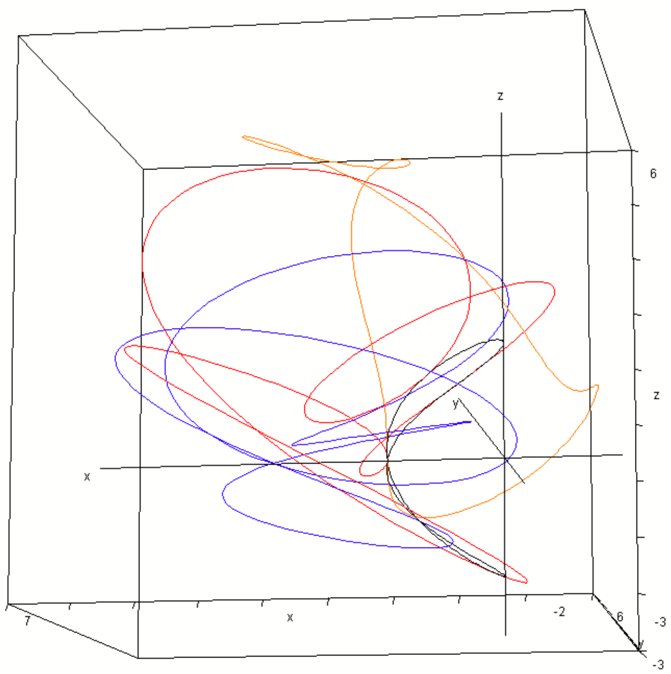
Screen shots with different values for the third coordinate of the pedal point.



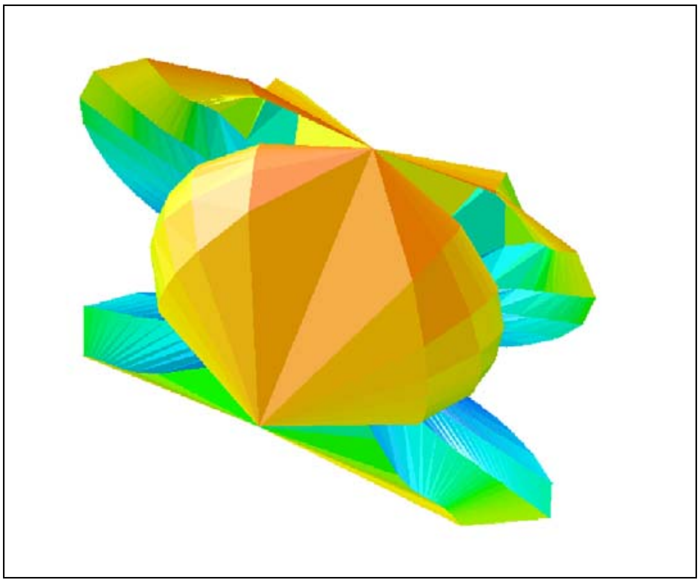


9 Appendix

The plot below shows the Window of Viviani (black), its pedal curve (red). the pedal curve with the normal instead of the tangent (blue) and the pedal curve with the binormal instead of the tangent (gold).



This is the representation of a pedal surface using DPGraph. The DERIVE results can directly be exported to DPGraph.



Other investigations might be:

Locus of the foot points of the perpendicular lines passing the pedal point with the rectifying plane (tangent & binormal), the osculating plane (tangent and principal normal) and the normal plane (principal normal and binormal).

10 References

Alfred Gray, *Modern Differential Geometry of Curves and Surfaces*, CRC

Vladimir Rovenski, *Geometry of Curves and Surfaces with MAPLE*, Birkhäuser

Philip Todd, *Pedal and Skew Pedal Curves of a Parabola,* The Journal of Symbolic Geometry, Volume 1, 2006

Roger C. Alperin, *A Grand Tour of Pedals of Conics*, Forum Geometricorum, Volume 4 (2004) 143–151.

https://en.wikipedia.org/wiki/Pedal_curve

https://en.wikipedia.org/wiki/Negative_pedal_curve

http://jwilson.coe.uga.edu/emt669/student.folders/jeon.kyungsoon/document1/Essay1.html

Eric W. Weisstein: *Pedal Curve,* http://mathworld.wolfram.com/PedalCurve.html

Eric W. Weisstein: *Contrapedal Curve,* http://mathworld.wolfram.com/ContrapedalCurve.html

Thanks, hope you enjoyed the

WONDERFUL WORLD OF PEDAL CURVES!

All files are available on request.