Investigating stars in 2D and 3D with DGS and CAS
(a model of research work at all levels)

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Abstract

This paper is the story of a research work mediated with technology and especially with DGS (TI N'Spire, Cabri 2 Plus and Cabri 3D) and CAS (TI N'Spire and Maple). It aims to show the principal stages of such an experimental process and some very important techniques of investigation. We will show the importance of collecting and interpreting data to increase the possibility to get some conjectures related to interesting relations between areas or volumes and to validate or invalidate them. Moving from 2D to 3D will give the opportunity to show the crucial role of generalization in a math research work. Changing the direction of a research can enrich the given problem and gives other perspectives of research. This paper gives also the opportunity to provide to teachers different examples of investigation that can be performed by students at different levels, from the level of middle school to the college level.

1. First investigations with circles constructed from regular polygons

1.1. Presentation of the problem

1.1.1. Initial constructions with TI N'Spire (or Cabri 2 Plus)

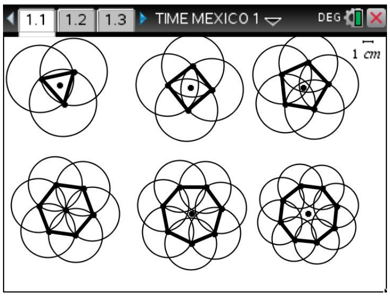


figure 1

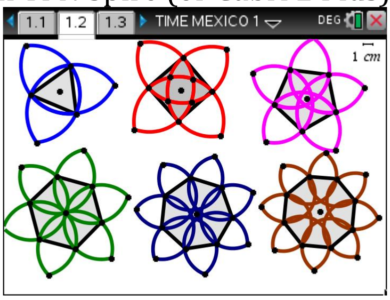


figure 2

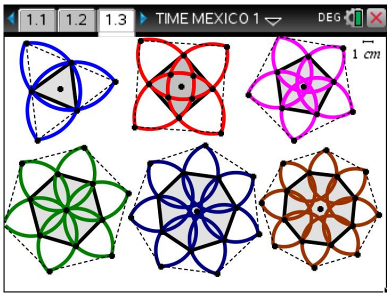


figure 3

We start with regular polygons (called n-gons, from an equilateral triangle to a regular octogon) as shown in figure 1. We can see in the same figure the n circles of these n-gons centred at each vertex and having as a radius the length of the side.

In figure 2, we focus our attention on some parts of these circles which are the bold arcs containing two exterior intersection points of the circles and the center of the n-gon. We can see n-stars corresponding to n-gons.

In figure 3, we focus our attention on the biggest n-stars constructed in figure 2. The vertices of these stars define another n-gon. It is easy to prove that these polygons are regular polygons

1.1.2. What do we want to do ?

We want to explore a possible relation between the area of the initial polygon and the final one. In order to find out a possible ratio we will experiment in using the measuring tools and the calculator of the software.

1.2. Collecting data and interpretation

1.2.1. Collecting data (figure 4)

For each case, we measure the areas of the two polygons, evaluate the ratio between the second one and the first one (in evaluating the expression $\frac{a}{b}$ created on the left side of figure 4) and display the results on the screen under each case. This table summarize the results got in experimenting like this.

In the first line the number n of sides of the initial and final polygon.
In the second line the ratio between the area of the final polygon and the initial polygon.

n	3	4	5	6	7	8
ratio	4	3.73	3.34	3	2.73	2.52

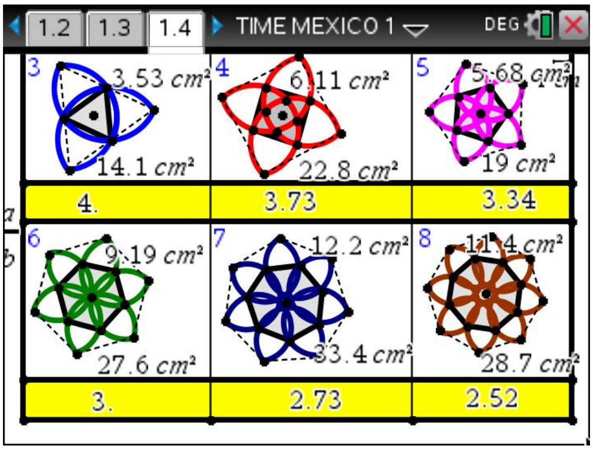


figure 4

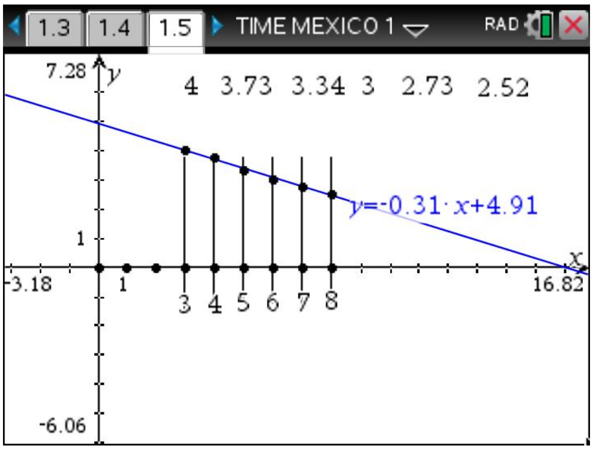


figure 5

1.2.2. First regression (figure 5)
The previous figures are constructed in a geometry page. Figure 5 is a figure of a Graphs & Geometry page. In this figure we have constructed the 6 points having as coordinates (n, *ratio*). Perceptively we can see that this set of points belongs approximatively to a line. So we have constructed a line having $y = ax+b$ as an equation, which allows us to translate and to rotate it until we obtain the superimposition of these points and and the line. Which is hidden behind that is the conjecture of a linear relation between these areas and the validation of this conjecture perceptively (praxeology G1).
A deductive approach of this figure (in the meaning of Duval) can help to invalidate this conjecture : in fact, this *ratio* is a positive number and cannot be negative for any n which is not the case in the right part of figure 5.
1.2.3. More data lead to abandon DGS and change the direction of the research
If we conduct the same experiment with other n-gons but with $n > 8$ (from $n = 9$ to 14), we obtain what you can see in figures 6, 7 and 8

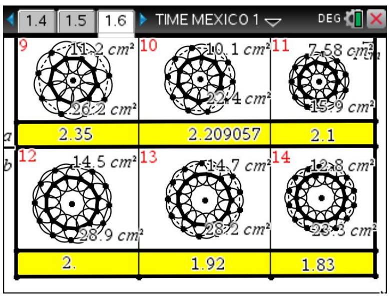


figure 6

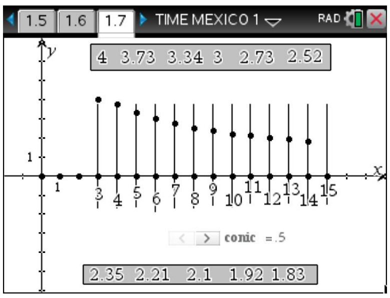


figure 7

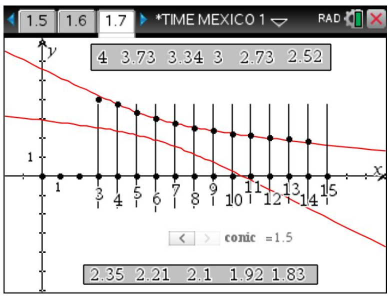


figure 8

Figure 6 shows the experiments conducted. The conjecture telling that there is a linear relation between the areas is definitely invalidated (perceptively : again G1) by figure 7. Another conjecture can appear when we construct the conic passing through 5 of the 12 points represented here. Perceptively again, it seems that this branch of hyperbola approximate this set of points. But like in the previous experiment, an expert approach of this graph and this conic which is a deductive approach leads to the rejection of this conjecture because this hyperbola will cross the x-axis which is impossible.

1.2.4. Other experiments with other regressions

To increase the accuracy of our regression let us use a table filled manually (figure 9) and its quick graph (figure 10). Then, we can try below, to approximate this graph with several regressions

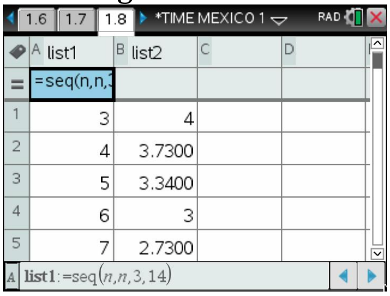


figure 9

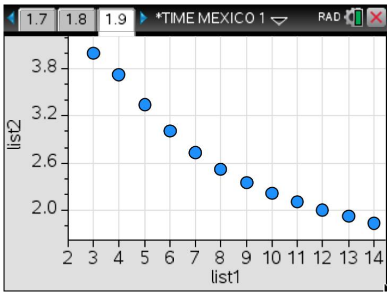


figure 10

Quadratic regression :
Done in figure 11 : not to bad.
But in figure 11, this approximation is rejected.

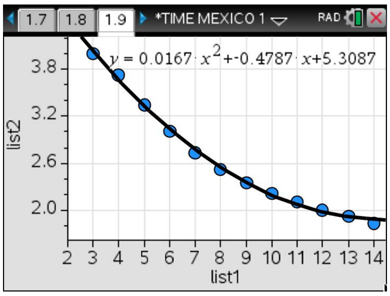


figure 11

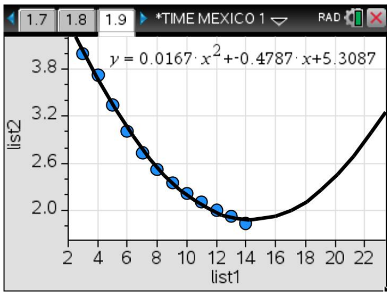


figure 12

Cubic regression :
Done in figure 12 : not to bad again.
But in figure 13, this approximation is rejected.

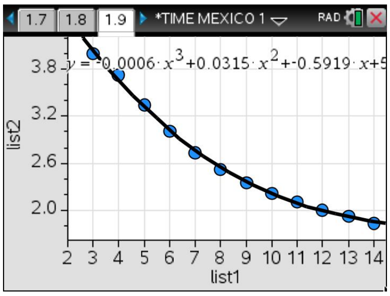


figure 12

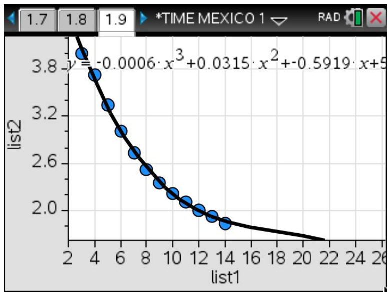


figure 13

Quartic regression :
Done in figure 14 : not to bad again.
But in figure 15, this approximation is rejected.

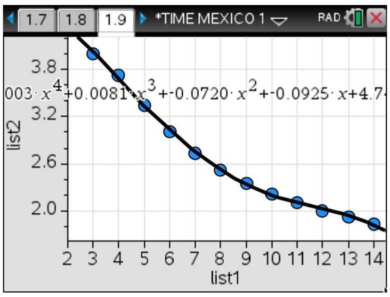


figure14

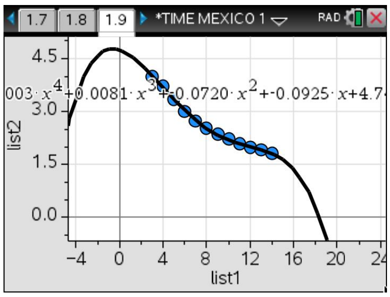


figure 15

The power, exponential and logarithmic regressions that we can perform, presented below, respectively in figures 16, 17 and 18, are, on the first glance, definitely not appropriate. It is why, at this stage of our work ,we decide to try to find out formally the possible relation between these areas.

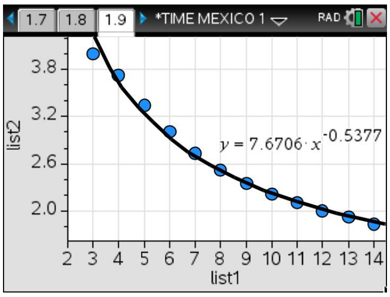


figure 16

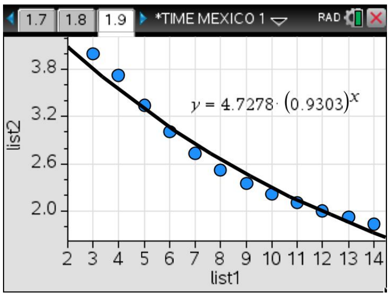


figure 17

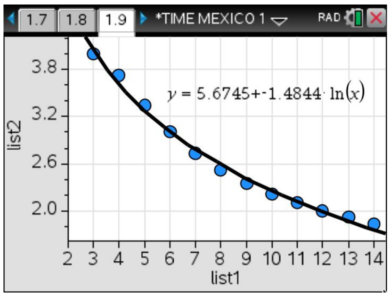


Figure 18

1.2.5. The formal proof

The experimental process we have conducted until now aimed to lead us to a conjecture easy to guess. As it was not the case and we think that we are skilled enough to discover this possible relation formally, we conduct the following reasoning.

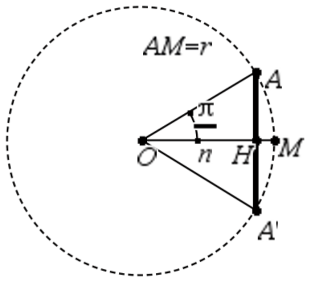


figure 19

We evaluate the area of an *n*-gon (regular polygon with *n* sides) inscribed in a circle having *r* as a radius. In figure 19, *AA'* is one of the side of the initial n-gon.

$OH = r.\cos(\frac{\pi}{n})$ and $AH = r.\sin(\frac{\pi}{n})$. Then, the area of triangle *OAA'* is given by : $r^2.\sin(\frac{\pi}{n}).\cos(\frac{\pi}{n})$. Finally the area of the n-gon is :

$$\boldsymbol{a} = \frac{nr^2}{2}\sin\left(\frac{2\pi}{n}\right)$$

Now we evaluate the area $\mathcal{A}$ of the biggest n-gon obtained by the previous constructions. We need to know the radius of the circle in which this n-gon is inscribed

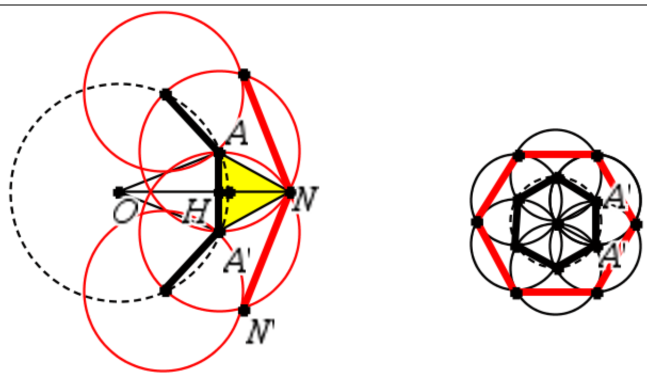


figure 20

In the right part of figure 20, we have represented the two n-gons (the initial and the final).
The left part of figure 20 helps us to evaluate *ON* which is the radius of the final n-gon and by the way its area $\mathcal{A}$.

$HN = AH.\tan\left(\frac{\pi}{3}\right) = r.\sqrt{3}.\sin(\frac{\pi}{n})$.

$ON = OH + HN = r.\cos(\frac{\pi}{n}) + r.\sqrt{3}.\sin(\frac{\pi}{n})$ and then :

$$\mathcal{A} = \frac{nr^2}{2}.\left(\cos(\frac{\pi}{n}) + \sqrt{3}.\sin(\frac{\pi}{n})\right)^2.\sin\left(\frac{2\pi}{n}\right)$$

Finally, the ratio we wanted to evaluate, $\mathcal{A}/\boldsymbol{a}$ is given by the formula :

$$\boxed{ratio = \left(\cos(\frac{\pi}{n}) + \sqrt{3}.\sin(\frac{\pi}{n})\right)^2}$$

Remarks :

- We can understand why all our attempts of regression failed.

- As $\lim_{n\to\infty}\left(\cos\left(\frac{\pi}{x}\right)+\sqrt{3}.\sin\left(\frac{\pi}{x}\right)\right)^2 = 1$, we can confirm the conjecture about the points of the graph approaching the x-axis when n is increasing..
- Finally we can confirm graphically (figure 21) the accuracy of this formula in displaying on the same system of axis, the curve of the fonction $ratio(x)$ defined by : ***ratio*(*x*) =** $\left(\mathbf{cos}\left(\frac{\pi}{x}\right)+\sqrt{3}.\mathbf{sin}\left(\frac{\pi}{x}\right)\right)^2$

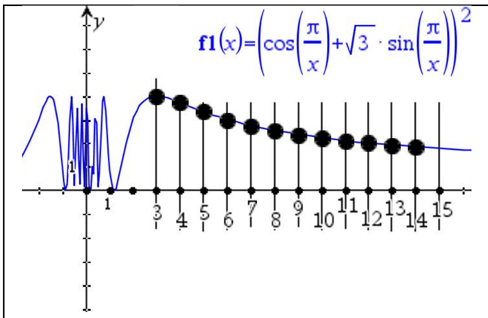


figure 21

In fact, we can check in figure 21, the superimposition of this curve with the graph representing the ratios obtained experimentally (we have worked on G2 for the proof and validated the result we obtained in G1 Informatique)

2. Second investigations with parabolas constructed from regular polygons (focusing on areas of polygons)

2.1. Presentation of the problem

In this part, we try to solve the same problem where the circles are replaced by parabolas having as a focus and vertex two consecutive vertices of the n-gon as shown for a 4-gon (square) in figures 22, 23 and 24. Parabolas 1 and 2 are constructed in figure 22. The four parabolas appear in figure 23. The final 4-gon is constructed in figure 24.

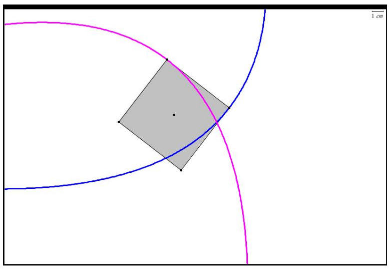


figure 22

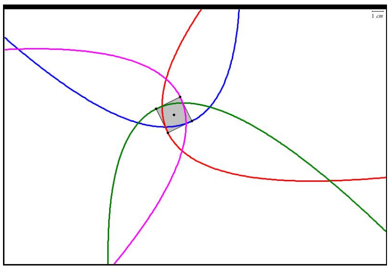


figure 23

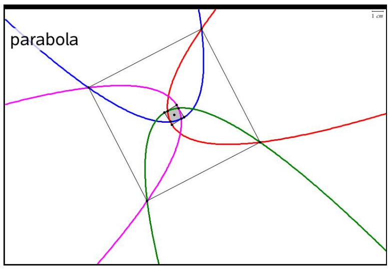


figure 24

In figure 24, we can see that the four parabolas can define a curvilinear square and two stars (4-stars) with four petals each. We will focus our attention on these figures in the next paragraph.

Here we conduct similar experiments to the ones conducted with the circles.

2.2. Constructing figures and collecting data

We experiment on n-gons from n = 3 to n = 11. The algorithm of construction is always the same. Start with a regular polygon with n sides. Construct the n parabolas associated to each couple of two consecutive vertices of this n-gon. Finally, construct the biggest polygon (which is a n-gon too) defined by the intersection points between these parabolas. Then we measure the areas of the two n-gons of each figure. Last but not the least, we evaluate and display the ratio of these two areas. This work is displayed below in figures 26 to 34. The biggest n-gon appears after zooming out because it is a lot more bigger than the n-gon we started from.

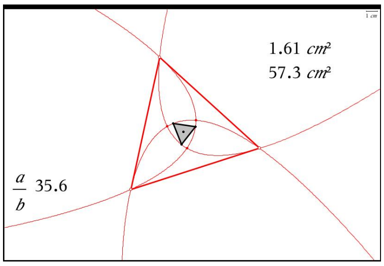


figure 26

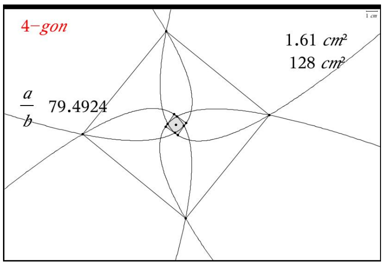


Figure 27

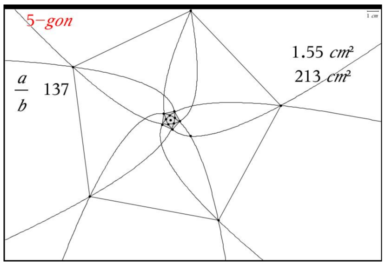


Figure 28

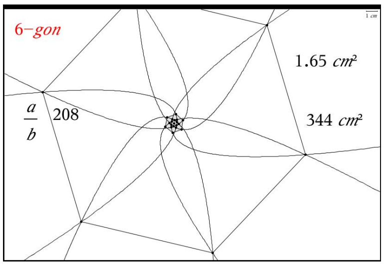


figure 29

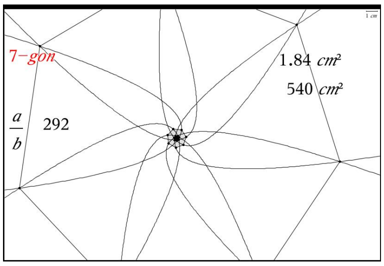


Figure 30

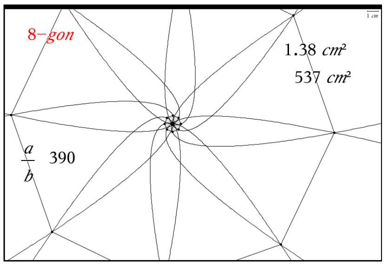


Figure 31

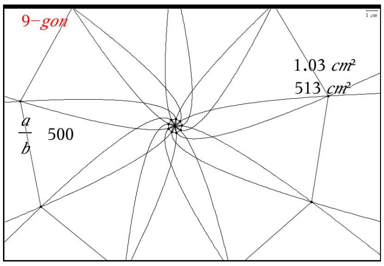


figure 32

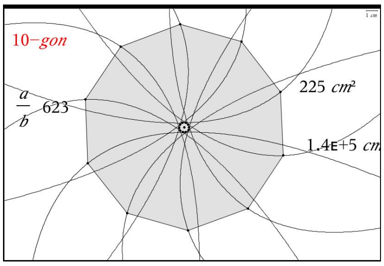


Figure 33

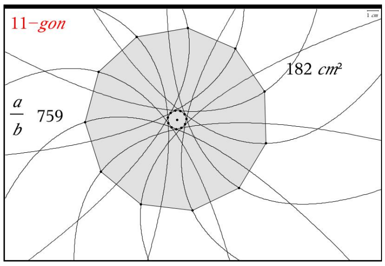


Figure 34

2.3. An attempt to approximate the points (n, ratio) with a quadratic fonction

We plot these points in the system of axis of a Graphs & Geometry page and we display the curve of the square function (figure 35). We deform this curve in translating it and changing its curvature which is possible in this environment in order to approximate better and better the previous points with a quadratic function (figure 36). The best we can do is displayed in figure 37.

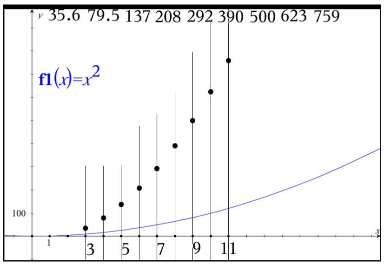


Figure 35

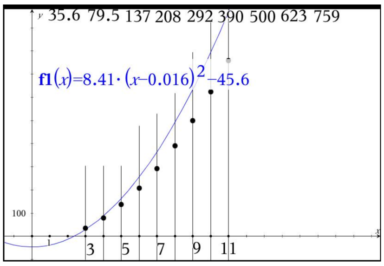


Figure 36

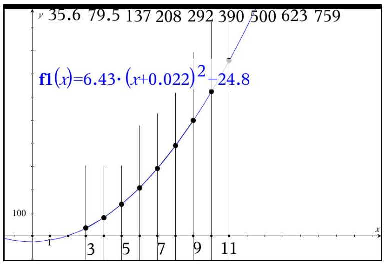


Figure 37

How can we use this last regression in order to generate a conjecture ? One way is to say : if there is a quadratic relation between n and ratio, this relation could be :
$ratio(n) = 6.5.n^2\text{-}25$.
Helas, ratio(11) = 761.5 which is not the ratio 759 obtained experimentally (see figure 34). Then this conjecture has to be rejected.

2.4. The special case of the nonagon

We know that for an hexagon, all the circles constructed for the problem of paragraph 1 have the center of this hexagon as a common point (figure 38). During the experiments conducted in paragraph 2 with parabolas, we had the opportunity, as shown in figure

39, to conjecture that for a regular nonagon (a 9-gon) all the parabolas have the center of this 9-gon as a common point. But if we zoom out, as done in figure 40, it is clear that this conjecture is false. Some measurements in this last figure lead to state that the ratio d/r where d is the distance between the center of the 9-gon and the vertex of one of the 9 parabolas and r the radius of the circle in which the 9-gon is inscribed is approximatively 1.6%.

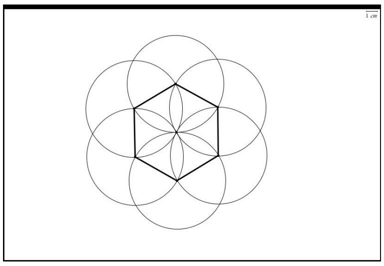


figure 38

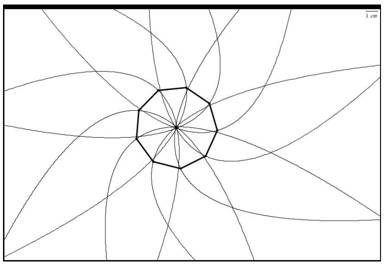


figure 39

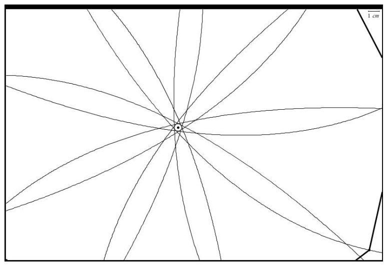


figure 40

2.5. An attempt for a formal proof

2.5.1. A reminder about cartesian and parametric equations of parabolas

In order to start any proof using coordinates, we need to use the following results (figures 41 and 42) where F is the focus of the parabola and (D) its directrix.

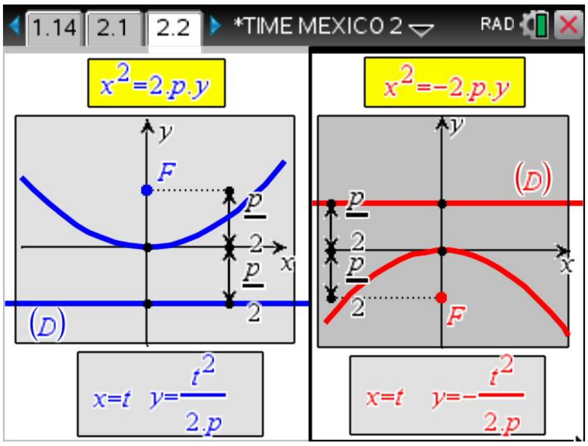


Figure 41

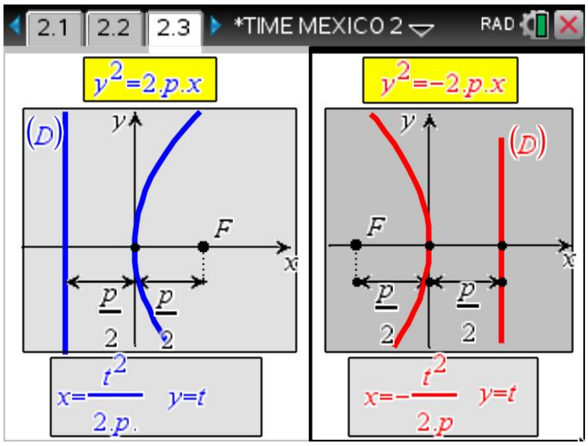


Figure 42

2.5.2. The formal proof with the technical power of the Note page

The principal issue is to evaluate the radius of the biggest star obtained with the construction with parabolas.

We check on figures 43 and 44 that each vertex of the biggest star is the intersection point between two consecutive parabolas which is not inside the initial n-gon (the first point can be seen in figure 43

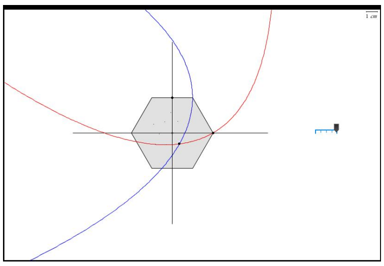


figure 43

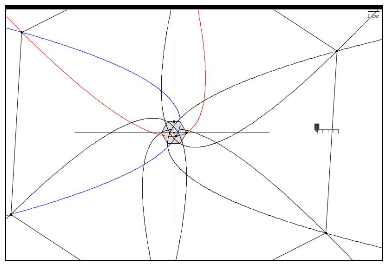


figure 44

So, let us evaluate the distance between the center of the n-gon and one of this vertex. We try to solve this problem analytically. So we work in a system of axis where the initial n-gon $A_1 \ldots A_n$ is constructed with A_1 and A_2 symmetric with respect to the y-axis.

We have represented in figure 45, A_1, A_2 and A_3, the three first points of the initial n-gon. As angle $\angle A_2OA_1 = \frac{2\pi}{n}$, the coordinates of A_2 are obtained easily and displayed on the right screenshot. Coordinates of A_1 can be obtained easily from the previous ones. In this figure, (P_1) and (P_2) are two consecutive parabolas. we will find out the coordinates of their intersection points.

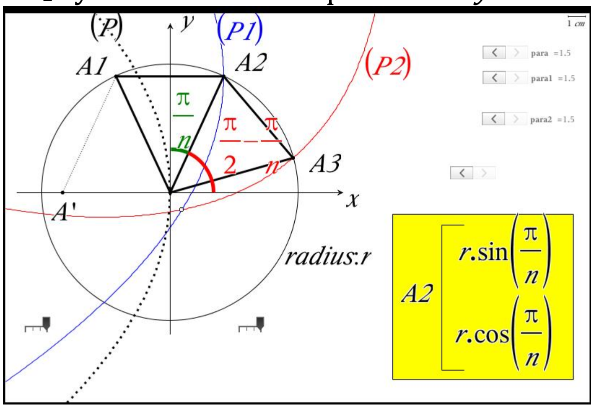


figure 45

In order to obtain the equations of (P_1) and (P_2), we use parabola (P) having A' as a focus and O as a vertex ($A'OA_2A_1$ is a parallelogram). That allows us to get the equation of (P) thanks to the previous reminder. The equations of (P_1) and (P_2) are obtained in using the translation transforming (P) in (P_1) (vector $\overrightarrow{OA_2}$) and the rotation centered in O transforming (P_1) in (P_2) (angle $\frac{2\pi}{n}$). We show below all what we did in a Note page of TI N'Spire (right column of the table below).

Evaluation of the parameter p of the three parabolas ->->->->->->->->	Common parameter p of the 3 parabolas (P), (P1) and (P2): $p = 2.A1A2 = 4.r.\sin\left(\frac{\pi}{n}\right)$ ....
Equation of (P) got with the previous reminder ->->->->->->->->	Equation of (P) : $y^2 = -2\cdot p\cdot x$ i.e. $y^2 = -8\cdot r.\sin\left(\frac{\pi}{n}\right)\cdot x$ ....
Cartesian equation of (P_1) got with the previous equation and the translation which vector is $\overrightarrow{OA_2}$ ->	Equation of (P1) : $\left(y - r.\cos\left(\frac{\pi}{n}\right)\right)^2 = -8\cdot r.\sin\left(\frac{\pi}{n}\right)\cdot\left(x - r.\sin\left(\frac{\pi}{n}\right)\right)$
Parametric equations of (P) got with the previous reminder ->->->	Parametric equations of (P) : $x = -\frac{t^2}{8\cdot r.\sin\left(\frac{\pi}{n}\right)}$, $y = t$ ....
Parametric equations of (P_1) got with the previous equation and the same translation : two functions depending of 3 parameters t, r and n created in Math boxes : it is why their names **xpta1** and **ypta1** are in bold in the Note page ->->->->->->->->->->->	Parametric equations of (P1) : $\mathbf{xpta1}(t,r,n) := \frac{-t^2}{8\cdot r\cdot\sin\left(\frac{\pi}{n}\right)} + r\cdot\sin\left(\frac{\pi}{n}\right)$ ▸ *Done* $\mathbf{ypta1}(t,r,n) := t + r\cdot\cos\left(\frac{\pi}{n}\right)$ ▸ *Done*

	r=rotation(O,$-\frac{2\cdot\pi}{n}$) mapping A1 onto A2
Creation in another Math box of the matrix **m** of the rotation r mapping A_1 onto A_2 ->->->->->->->	m=matrix of r $\mathbf{m}:=\begin{bmatrix}\cos\left(\frac{-2\cdot\pi}{n}\right) & -\sin\left(\frac{-2\cdot\pi}{n}\right)\\ \sin\left(\frac{-2\cdot\pi}{n}\right) & \cos\left(\frac{-2\cdot\pi}{n}\right)\end{bmatrix} \blacktriangleright \begin{bmatrix}\cos\left(\frac{2\cdot\pi}{n}\right) & \sin\left(\frac{2\cdot\pi}{n}\right)\\ -\sin\left(\frac{2\cdot\pi}{n}\right) & \cos\left(\frac{2\cdot\pi}{n}\right)\end{bmatrix}$ ....
Still in a math box creation of the matrix **m1** of the previous coordinates.	$\mathbf{m1}:=\begin{bmatrix}\mathbf{xpta1}(t,r,n)\\ \mathbf{ypta1}(t,r,n)\end{bmatrix} \blacktriangleright \begin{bmatrix}\sin\left(\frac{\pi}{n}\right)\cdot r-\frac{t^2}{8\cdot\sin\left(\frac{\pi}{n}\right)\cdot r}\\ \cos\left(\frac{\pi}{n}\right)\cdot r+t\end{bmatrix}$
Creation of matrix **m2** of the coordinates of a point of (P_2) as the image of a point of (P_1) by r. Evaluation of **m2 :=m.m1** ->->->->	$\mathbf{m2}:=\mathbf{m}\cdot\mathbf{m1}$ $\blacktriangleright \begin{bmatrix}\frac{8\cdot\sin\left(\frac{\pi}{n}\right)\cdot\left(\sin\left(\frac{\pi}{n}\right)\cdot\cos\left(\frac{2\cdot\pi}{n}\right)+\cos\left(\frac{\pi}{n}\right)\cdot\sin\left(\frac{2\cdot\pi}{n}\right)\right)\cdot r^2+8\cdot}{8\cdot\sin\left(\frac{\pi}{n}\right)\cdot r}\\ \frac{4\cdot\cos\left(\frac{\pi}{n}\right)\cdot\left(\cos\left(\frac{2\cdot\pi}{n}\right)-2\cdot\left(\sin\left(\frac{\pi}{n}\right)\right)^2\right)\cdot r^2+4\cdot c}{4\cdot r}\end{bmatrix}$
Transformation of matrix **m2** onto a list **n2** ->->->->->->->->->->->->->	$\mathbf{n2}:=\text{mat}\blacktriangleright\text{list}(\mathbf{m2})$ $\blacktriangleright \left\{\frac{8\cdot\sin\left(\frac{\pi}{n}\right)\cdot\left(\sin\left(\frac{\pi}{n}\right)\cdot\cos\left(\frac{2\cdot\pi}{n}\right)+\cos\left(\frac{\pi}{n}\right)\cdot\sin\left(\frac{2\cdot\pi}{n}\right)\right)\cdot r^2+8\cdot\sin}{8\cdot\sin\left(\frac{\pi}{n}\right)\cdot r}\right.$ ....
The parametric equations of (P_2) are the 1st and 2nd term of **n2** ->->	Parametric equations of (P2) $\mathbf{xpta2}(t,r,n):=\mathbf{n2}[1] \blacktriangleright \mathit{Done}\ \mathbf{ypta2}(t,r,n):=\mathbf{n2}[2] \blacktriangleright \mathit{Done}$ ....
To get the intersection points between (P_1) and (P_2), we substitute in the cartesian equation of (P_1), the parametric equations of (P_2) and we solve it with respect to *t*. And it does not work : to hard for the software -> We try to simplify the equation to solve in replacing *r* by 1 and *n* by 4 (case of the square) and it does not work again	Intersection between (P1) and (P2): $\text{solve}\left(\left(\mathbf{yPTA2}(t,r,n)-r.\cos\left(\frac{\pi}{n}\right)\right)^2=-8\cdot r.\sin\left(\frac{\pi}{n}\right)\cdot\left(\mathbf{xPTA2}(t,r,n)-r.\sin\left(\frac{\pi}{n}\right)\right),t\right)$ Syntax error, so, in a particular case (…) $\text{solve}\left(\left(\mathbf{ypta2}(t,1,4)-\cos\left(\frac{\pi}{4}\right)\right)^2=-8\cdot\sin\left(\frac{\pi}{4}\right)\cdot\left(\mathbf{xpta2}(t,1,4)-\sin\left(\frac{\pi}{4}\right)\right),t\right)$ Does not work…

At this stage we could abandon our attempt to solve this problem but we will nevertheless try to solve the last particular case (square) in changing the technique of the proof

2.5.3. The formal proof for the square

We come back to a 4-gon (a square) and put it in the system of axis represented in figure 46. Now we consider the two consecutive parabolas (P_1) (focus A_1 and vertex A_2) and (P_2) (focus A_2 and vertex O). We will easily find the equations of these parabolas thanks to the reminder and a translation.

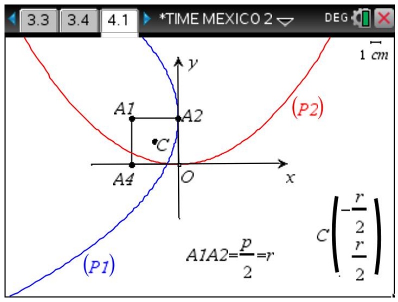


figure 46

Below are displayed the screenshots of the Note page where we solved the problem :

Cartesian equations of (P_1) and (P_2) thanks to the reminder ->->	Equation of (P1): $(y-r)^2=-4\cdot r\cdot x$ Equation of (P2): $x^2=4\cdot r\cdot y$ Intersection between (P1) and (P2):
The list of the abscissas of the intersection points is stored in **s1**	$\mathbf{s1}:=\text{zeros}\left(\left(\frac{x^2}{4\cdot r}-r\right)^2+4\cdot r\cdot x,x\right) \blacktriangleright \{-0.242692\cdot r,-4.59974\cdot r\}$
The second abscissa (the negative one) is stored in **xp** as **s1[2]**. It is the abscissa we want to find. Its ordinate is evaluated and stored in **yp**. **d** is the distance between C and this point. **dd** is the diameter of the biggest square. **side** is the length of the side of this square. **area** is its area. Finally, **ratio** is the ratio we wanted to evaluate between the original biggest square and the first one ->->->->	$\mathbf{xp}:=\mathbf{s1}[2] \blacktriangleright -4.59974\cdot r$ $\mathbf{yp}:=\frac{\mathbf{xp}^2}{4\cdot r} \blacktriangleright 5.2894\cdot r$ ⚠ $\mathbf{d}:=\sqrt{\left(\mathbf{xp}+\frac{r}{2}\right)^2+\left(\mathbf{yp}-\frac{r}{2}\right)^2} \blacktriangleright 6.30446\cdot \lvert r\rvert$ $\mathbf{dd}:=2\cdot \mathbf{d} \blacktriangleright 12.6089\cdot \lvert r\rvert$ $\mathbf{side}:=\frac{\mathbf{dd}}{\sqrt{2}} \blacktriangleright 8.91585\cdot \lvert r\rvert$ $\mathbf{area}:=\mathbf{side}^2 \blacktriangleright 79.4924\cdot r^2$ $\mathbf{ratio}:=\frac{\mathbf{area}}{r^2} \blacktriangleright 79.4924$ ⚠

The solution we have obtained is a solution in the approx mode. It is possible to chose in the settings of the document the exact mode and the Note Page is refreshed as shown below.

	Equation of (P1): $(y-r)^2=-4\cdot r\cdot x$ Equation of (P2): $x^2=4\cdot r\cdot y$ Intersection between (P1) and (P2): $\mathbf{s1}:=\text{zeros}\left(\left(\frac{x^2}{4\cdot r}-r\right)^2+4\cdot r\cdot x,x\right) \blacktriangleright \{\square\}$ ⚠ $\mathbf{xp}:=\mathbf{s1}[2] \blacktriangleright$ Error: Dimension error $\mathbf{yp}:=\frac{\mathbf{xp}^2}{4\cdot r} \blacktriangleright \frac{33852163866580478449126681\cdot r}{6400000000000000000000000}$ ⚠

	$\mathbf{d} := \sqrt{\left(\mathbf{xp} + \frac{r}{2}\right)^2 + \left(\mathbf{yp} - \frac{r}{2}\right)^2}$ $\blacktriangleright \frac{\sqrt{16280052194487637553406163406975955865452514607761} \cdot \lvert r \rvert}{6400000000000000000000000}$ $\mathbf{dd} := 2 \cdot \mathbf{d}$ $\blacktriangleright \frac{\sqrt{16280052194487637553406163406975955865452514607761} \cdot \lvert r \rvert}{3200000000000000000000000}$ $\mathbf{side} := \frac{\mathbf{dd}}{\sqrt{2}}$ $\blacktriangleright \frac{\sqrt{32560104388975275106812326813951911730905029215522} \cdot \lvert r \rvert}{6400000000000000000000000}$ $\mathbf{area} := \mathbf{side}^2$ $\blacktriangleright \frac{16280052194487637553406163406975955865452514607761 \cdot r^2}{204800000000000000000000000000000000000000000000}$ $\mathbf{ratio} := \frac{\mathbf{area}}{r^2}$ $\blacktriangleright \frac{16280052194487637553406163406975955865452514607761}{204800000000000000000000000000000000000000000000}$ ⚠

Eventually, we are successful in changing only in the CAS of TI N'Spire the way we organize this proof.

3. Third investigations with parabolas constructed from regular polygons (focusing on areas of stars and some ratios)

3.1. Presentation of the problem

In this part we will focus our attention only on two cases, the case of the 3-gon (equilateral triangle) and the case of the 4-gon (square) and especially on the curvilinear regular polygons (equilateral triangle or square) and stars defined by the parabolas. Now the problem to solve will be to determining the ratio between the area of the curvilinear polygon or the star and the area of the initial polygon.
Let us display all these cases before solving this problem.

Case of the 3-gon

The original 3-gon is triangle *A1A2A3*. With the 3 parabolas, we create the blue curvilinear triangle *B1B2B3* (figure 47) and the red star *C1B1C2B2C3B3* (figure 48).

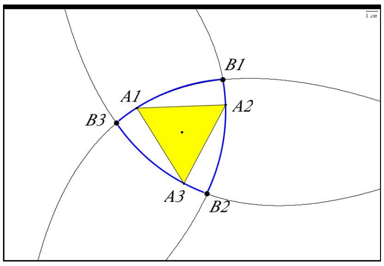


figure 47

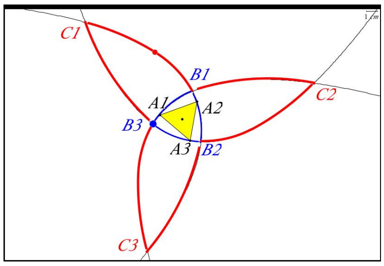


figure 48

Case of the 4-gon

In figures 49, 50 and 51, from the original 4-gon (square *A1A2A3A4*), we can create with the four parabolas a curvilinear square B1B2B3B4, an intermediate star with four petals *C1B1C2B2C3B3C4B4* and the biggest star with four petals too *D1C1D2C2D3C3D4C4*. From figure 49 to figure 51, we have only zoomed out.

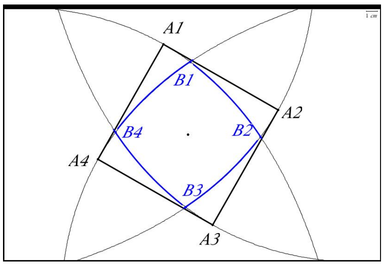


figure 49

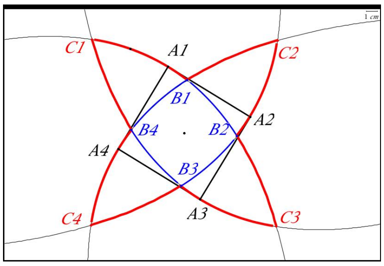


figure 50

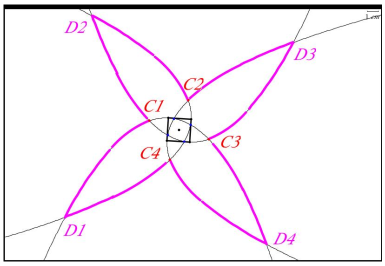


figure 51

3.2. Experiments starting from a 3-gon

3.2.1. Case of the curvilinear triangle

In a Geometry page (figure 52)

We create a polygon to approximate the area of the curvilinear triangle ABM which is *ytr(AMC)* (or y).

We display the area of the initial 3-gon ABC which is *gtr(ABC)* (or g).

A good approximation of the area of the curvilinear triangle MNP is given by the value of the expression $g+3.y$.

The ratio between these areas is evaluated and displayed as v1 (2.26)

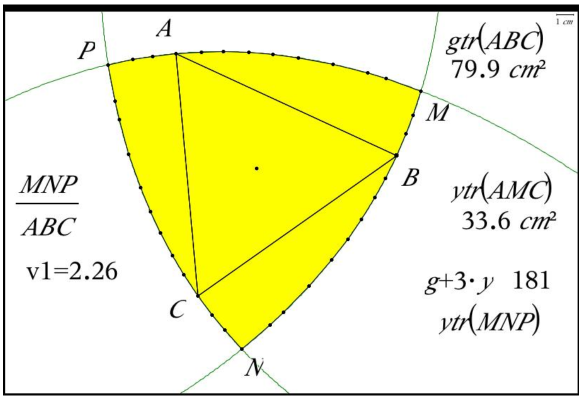


figure 52

3.2.2. Case of the two 3-Stars

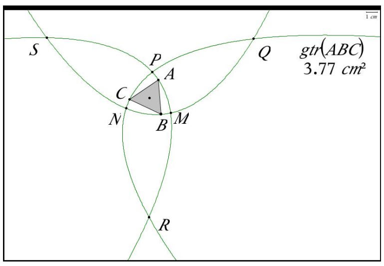


Figure 53

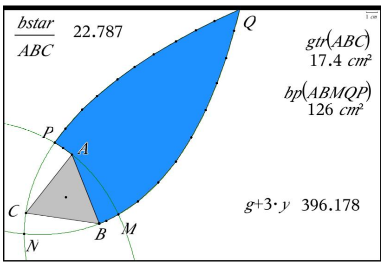


Figure 54

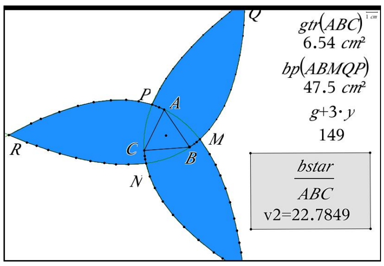


Figure 55

In a Geometry page (figures from 53 to 55),

We display the star with three petals limited by Q, R and S (figure 53) and the area of the initial 3-star *gtr(ABC)* (or g)

We create in figure 54 a polygon to approximate the area of the blue polygon $APQMB$ which is *bp(ABMPQ)* (or y).

In figure 55, we rotate this first blue petal twice around the center of the 3-gon (angle $\frac{2\pi}{3}$) to complete the initial blue polygon and obtain the entire blue polygon. An approximation of the area of this star is displayed as *bstar*.

The ratio between these areas is evaluated and displayed as v2 (22.7849…).

Remark : it is strange that the value of $\frac{v2}{v1}$ is close to 10 (10.0737…). It means that the area of the blue star is approximatively 10 times the area of the curvilinear triangle. Can this ratio be equal exactly to 10 ? This is an open problem.

3.3. Experiments starting from a 4-gon

3.3.1. Case of the curvilinear square

Here (figure 56), we start from a square *ABCD* and the four parabolas having respectively two consecutive vertices as focus and vertex. We display its area (*sq(ABCD) or g*) We can see that these parabolas define inside this square a curvilinear square. We try to evaluate a good approximation of the area of this curvilinear square. But now we improve the way of constructing the polygon approximating it.

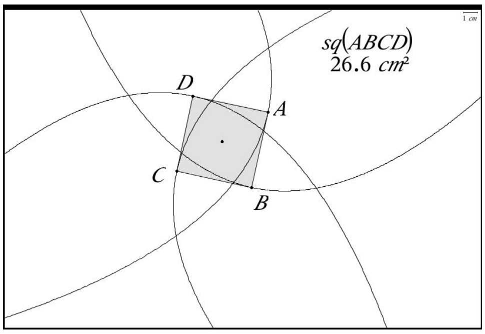


figure 56

Below in figure 57, we show how to construct on one of the sides of the curvilinear square a set of points which seem regularly plotted (using the tools *midpoint*, *ray* and *intersection point*).
In figure 58, after displaying number 90, we construct three other sets on points on the other sides of the curvilinear square. To obtain these other sets, we use three times the rotation centered at the center of the initial square and having 90° as an angle.
In figure 59, we create the yellow polygon passing through the verticices *M*, *N*, P and *Q* of the curvilinear square and the four sets of points previously constructed. We display it area (*ysq(MNPQ)* or *y*). Eventually we evaluate and display the ratio *g* divided by *y* as *gony* (1.4913). We will see below, after conducting an analytic reasooning for the formal proof that this approximation is a very good one.

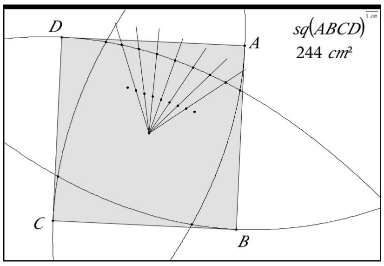


figure 57

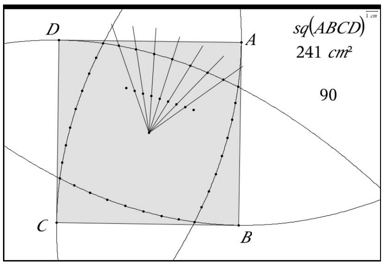


figure 58

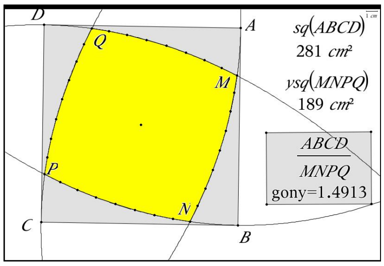


figure 59

3.3.2. Case of the first 4-star (corresponding to figure 50)

We use a similar technique to evaluate an approximation of the area of this star and eventually the ratio between this area and the area of the initial grey square.
In figures 60, 61 and 62, we can see the different steps of this experiment leading to an experimental approximation of the ratio (bong) between the area of the blue star (bluestar or b) and the area of the initial square which is 2.4079…

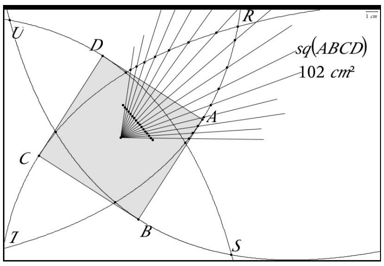


figure 60

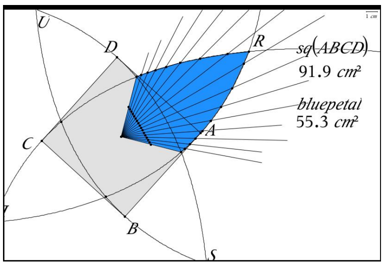


figure 61

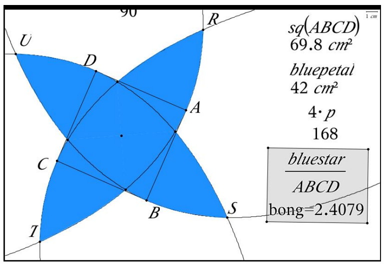


figure 62

3.3.3. Case of the second 4-star (corresponding to figure 51)

Same way of experimenting to get finally the ratio (pong) between the area of the purple star (*purpstar* or p) and the area of the initial square which is 34.4…

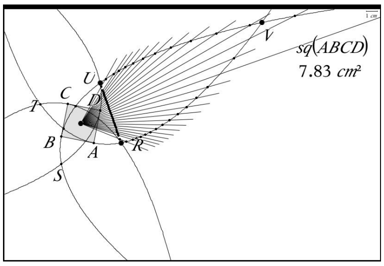


figure 63

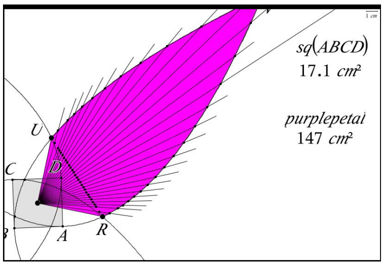


figure 64

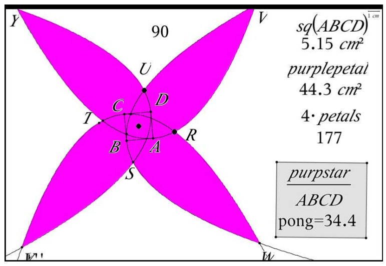


figure 65

3.4. What we have got experimentally

Here are the ratios we got : *gony* ≈1.4913, *bong*≈2.4079 and *pong*≈34.4.

Thanks to these results we can deduce the following ratios :

Ratio between the area of the first 4-star and the area of the curvilinear square **:** *bong***gony*≈3.59091

Ratio between the area of the second 4-star and the area of the curvilinear square **:** pong*gony≈51.3186.

About *gony*, it would be possible to conjecture that this ratio is exactly 1.5. In reality, as *g* is the exact value of the area of the initial square and *y* is an approximation which is less than the exact value of the area of the curvilinear square, we can say definitely that *gony* is less than 1.4913.

3.5. Formal proof for the curvilinear square (figure 49)

We chose for this proof to construct the square and the parabolas as it is shown in figure 66. *M* and *M'* are the two intersection points between parabola (*P1*) and parabola (*P2*). To solve the problem, we need to find the coordinates of point M. In figure 67, we have displayed the coordinates of these two points given by the sofware.

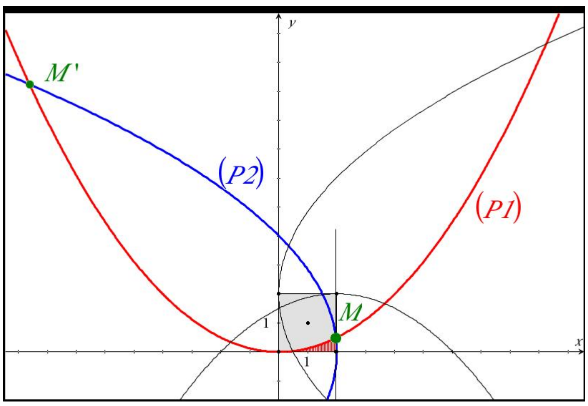


Figure 66

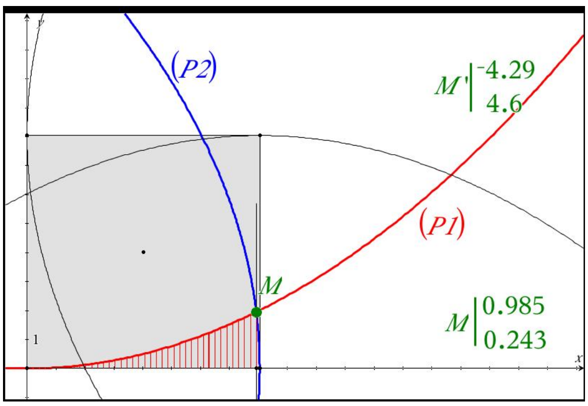


Figure 67

In order to evaluate the area of the curvilinear square, we will substract 4 times the red area to the area of the initial square. This work is shown in the screenshot of the Note page below.

-Equations of (*P1*) and (*P2*) (*a* is the side of the initial square)	$y=\frac{1}{4\cdot a}x^2$ and $x=\frac{-1}{4\cdot a}y^2+a$
-The two ordinates of their two intersection points are stored in list **cc** ->->->->->	**cc**:=zeros$\left(y-\frac{1}{4\cdot a}\cdot\left(\frac{-1}{4\cdot a}\cdot y^2+a\right)^2,y\right)$ ▸ $\{4.59974\cdot a,0.242692\cdot a\}$
The second term of this list, the ordinate of *M*, is stored in **ycc**	**ycc**:=**cc**[2] ▸ $0.242692\cdot a$
-The abscissa of M is evaluated with the equation of (*P2*) and stored in **xcc**	**xcc**:=$\frac{-1}{4\cdot a}\cdot$ **ycc**^2+a ▸ $0.985275\cdot a$ ⚠
-itg1 is the area between the x-axis, (*P1*), *O* and *M* ->	**itg1**:=$\int_0^{\mathbf{xcc}}\left(\frac{1}{4\cdot a}\cdot x^2\right)dx$ ▸ $0.079706\cdot a^2$
-itg2 is the area between the x-axis, (*P2*), *M* and the vertex of (*P2*) ->->->->->->	**itg2**:=$\int_{\mathbf{xcc}}^{a}\sqrt{4\cdot a\cdot(a-x)}\,dx$ ▸ $0.002382\cdot a\cdot\|a\|$
-curvi is the red area ->->->	**curvi**:=$a^2-4\cdot($**itg1**+**itg2**$)$ ▸ $-0.00953\cdot a\cdot(\|a\|-71.4801\cdot a)$
r1 is the ratio between the curvilinear square and the initial one. r2 is its inverse and the ratio we have evaluated experimentally with 1.49..	**r1**:=**curvi**$\|a=1$ ▸ 0.671646 **r2**:=$\frac{1}{\mathbf{r1}}$ ▸ 1.48888

3.6. Formal proof for the curvilinear triangle (figure 47)

In reality, this problem was the starting point of this research work. As we did in 3.5. we will use the CAS of TI N'Spire in a Note page to solve it formally.

As the ratio we want to evaluate doest depend of the size of the initial 3-gon (equilateral triangle), we chose a triangle in which each side measures 1 unit. We construct this triangle as done in figure 68. We have principally to evaluate the area of the curvilinear triangle *OAI*. To get the area of the curvilinear triangle we are interested with, we multiply this area by 3 before adding the area of the 3-gon.

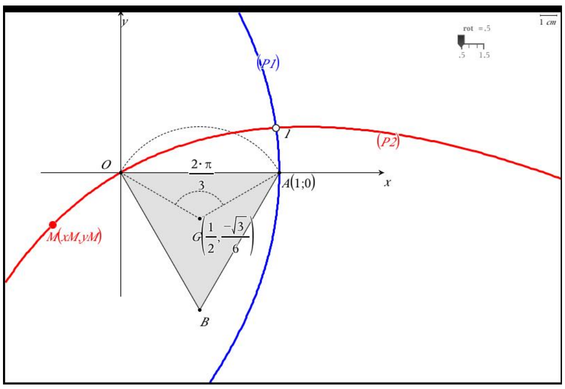


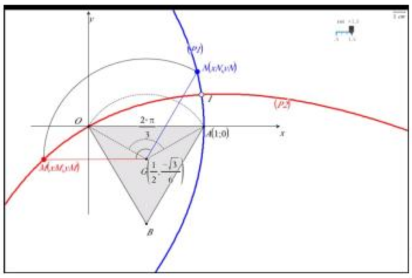
figure 68

Now we display the screenshots of the Note page where we solve the problem.

Equation of (P1): $y^2=-4\cdot(x-1) \Leftrightarrow y=\sqrt{-4\cdot(x-1)}$ or $y=-\sqrt{-4\cdot(x-1)}$

....

r=rotation(G; $\frac{-2\cdot\pi}{3}$) mapping (P2) onto (P1)



q= matrix of r

$$\mathbf{q}:=\begin{bmatrix}\cos\left(\frac{-2\cdot\pi}{3}\right) & -\sin\left(\frac{-2\cdot\pi}{3}\right)\\ \sin\left(\frac{-2\cdot\pi}{3}\right) & \cos\left(\frac{-2\cdot\pi}{3}\right)\end{bmatrix} \blacktriangleright \begin{bmatrix}\frac{-1}{2} & \frac{\sqrt{3}}{2}\\ \frac{-\sqrt{3}}{2} & \frac{-1}{2}\end{bmatrix}$$

....

Equation of (P2):

M(x,y)∈(P2)⇔N=r(M)∈(P1). As Vect(GN)=r(Vect(GM)), N= G+r(Vect(GM))

$$\mathbf{gm}:=\begin{bmatrix}xm-\frac{1}{2}\\ ym+\frac{\sqrt{3}}{6}\end{bmatrix} \blacktriangleright \begin{bmatrix}xm-\frac{1}{2}\\ ym+\frac{\sqrt{3}}{6}\end{bmatrix}$$

$$\mathbf{gn}:=\mathbf{q}\cdot\mathbf{gm} \blacktriangleright \begin{bmatrix}\frac{-xm}{2}+\frac{ym\cdot\sqrt{3}}{2}+\frac{1}{2}\\ \frac{-xm\cdot\sqrt{3}}{2}-\frac{ym}{2}+\frac{\sqrt{3}}{6}\end{bmatrix}$$

$$\text{coordG}:=\begin{bmatrix}\frac{1}{2}\\ -\frac{\sqrt{3}}{6}\end{bmatrix} \blacktriangleright \begin{bmatrix}\frac{1}{2}\\ \frac{-\sqrt{3}}{6}\end{bmatrix}$$

$$\mathbf{coordn}:=\mathbf{gn}+\mathbf{coordg} \blacktriangleright \begin{bmatrix}\frac{-xm}{2}+\frac{ym\cdot\sqrt{3}}{2}+1\\ \frac{-xm\cdot\sqrt{3}}{2}-\frac{ym}{2}\end{bmatrix}$$

$$\mathbf{lcoordn}:=\text{mat}\blacktriangleright\text{list}(\mathbf{coordn}) \blacktriangleright \left\{\frac{-xm}{2}+\frac{ym\cdot\sqrt{3}}{2}+1,\frac{-xm\cdot\sqrt{3}}{2}-\frac{ym}{2}\right\}$$

$$\mathbf{xn}:=\mathbf{lcoordn}[1] \blacktriangleright \frac{-xm}{2}+\frac{ym\cdot\sqrt{3}}{2}+1$$

$$\mathbf{yn}:=\mathbf{lcoordn}[2] \blacktriangleright \frac{-xm\cdot\sqrt{3}}{2}-\frac{ym}{2}$$

....

(xm,ym)∈(P2) ⇔ (xn,yn) ∈(P1) ⇔ $\mathbf{yn}^2+4\cdot(\mathbf{xn}-1)$=0

$\text{solve}\left(\mathbf{yn}^2+4\cdot(\mathbf{xn}-1)=0,ym\right)$

$\blacktriangleright\ ym=-\left(4\cdot\sqrt{2\cdot xm+3}+(xm+4)\cdot\sqrt{3}\right)$ or $ym=4\cdot\sqrt{2\cdot xm+3}-(xm+4)\cdot\sqrt{3}$

Conclusion : (P2) can be interpreted as the union of the curves of the two functions f and g defined by $y= f(x) = -(x+4).\sqrt{3}$ $+4\sqrt{2x+3}$ and $y= g(x) = -(x+4).\sqrt{3}$ $-4\sqrt{2x+3}$
We can check it in the two figures 69 and 70
In figure 69 who have zoomed out to see better the shape of (P2)
In figure 70 we have displayed the curve of the two functions *f* and *g* (the first one in blue dotted and second one in green dotted. These two functions are defined for $x \geq -\frac{3}{2}$.

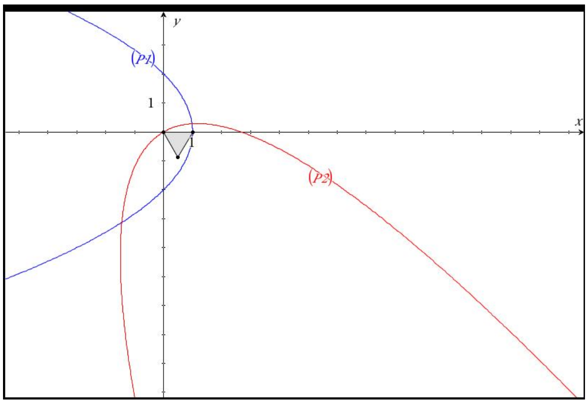


figure 69

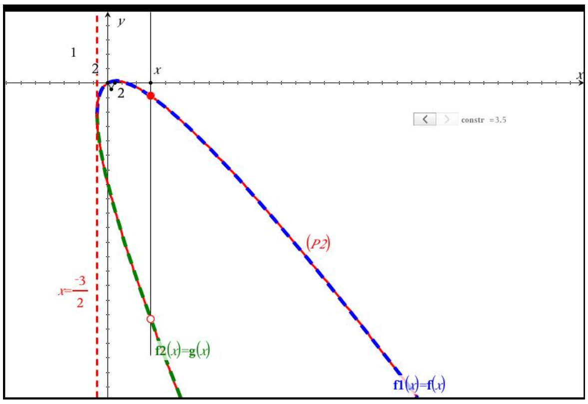


figure 70

Definitely, *f* is the function that will be use later to evaluate the first part of the curvilinear triangle *OAI*.

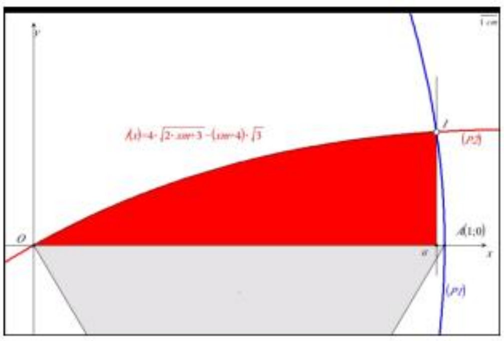
$\mathbf{f}(xm)$ ▸ $4\cdot\sqrt{2\cdot xm+3}-(xm+4)\cdot\sqrt{3}$

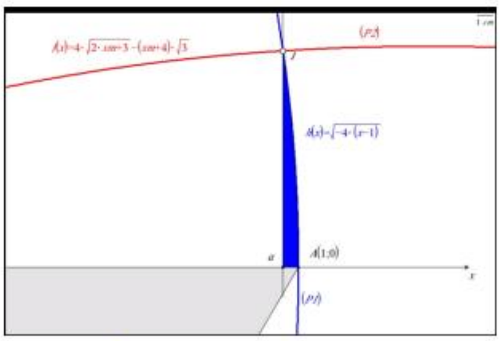
solve$\left(\mathbf{f}(x)=\sqrt{-4\cdot(x-1)},x\right)$ ▸ $x=0.980005$

aa:=zeros$\left(\mathbf{f}(x)-\sqrt{-4\cdot(x-1)},x\right)$ ▸ $\{0.980005\}$

a:=**aa**[1] ▸ 0.980005

a ▸ 0.980005





$\mathbf{itg1}:=\int_0^{\mathbf{a}} \mathbf{f}(x)\,dx$ ▸ 0.179021

$\mathbf{itg2}:=\int_{\mathbf{a}}^{1} \sqrt{-4\cdot(x-1)}\,dx$ ▸ 0.00377

So the end of the reasoning can be conducted like this :

Area of the triangle *OAI* : **add** ->->	$\mathbf{add}:=\mathbf{itg1}+\mathbf{itg2} \blacktriangleright 0.182791$
Area of the initial 3-gon : **tr** ->->->->->->->->	$\mathbf{tr}:=\frac{\sqrt{3}}{4} \blacktriangleright \frac{\sqrt{3}}{4}$
Area of curvilinear triangle : **curvtr** ->->->	$\mathbf{curvtr}:=\mathbf{tr}+3\cdot \mathbf{add} \blacktriangleright 0.981385$
Expected ratio : **ratio** ->->->->->->->->->->->	$\mathbf{ratio}:=\frac{\mathbf{curvtr}}{\mathbf{tr}} \blacktriangleright 2.26641$
With more digits ->->->->->->->->->->->->	$\mathbf{ratio} \blacktriangleright 2.266411142$ 2.266411142388979451

This result proves that our experiment was conducted correctly and that the software is very reliable. In fact, the ratio obtained experimentally in 3.2.1. was 2.26…

Remark : this Note page can give the exact result of our problem if we move the settings from the **approx** or **auto** to the **exact** mode as shown below :

$$\mathbf{itg1}:=\int_0^{\mathbf{a}} \mathbf{f}(x)\, dx$$

$$\blacktriangleright \frac{41333409837559\cdot\sqrt{124000229512677}}{31250000000000000000}-\frac{42001114297365822068903706329\cdot\sqrt{3}}{5000000000000000000000000000}$$

$$\mathbf{itg2}:=\int_{\mathbf{a}}^{1} \sqrt{-4\cdot(x-1)}\, dx \blacktriangleright \frac{999770487323\cdot\sqrt{1999540974646}}{375000000000000000000}$$

....

$$\mathbf{add}:=\mathbf{itg1}+\mathbf{itg2} \blacktriangleright \frac{41333409837559\cdot\sqrt{124000229512677}}{31250000000000000000}+\frac{999770487323\cdot\sqrt{1999540974646}}{375000000000000000000}-\frac{42001114297365822068903706329\cdot\sqrt{3}}{5000000000000000000000000000}$$

$$\mathbf{tr}:=\frac{\sqrt{3}}{4} \blacktriangleright \frac{\sqrt{3}}{4}$$

$$\mathbf{curvtr}:=\mathbf{tr}+3\cdot \mathbf{add}$$

$$\blacktriangleright \frac{124000229512677\cdot\sqrt{124000229512677}}{31250000000000000000}+\frac{999770487323\cdot\sqrt{1999540974646}}{125000000000000000000}-\frac{124753342892097466206711118987\cdot\sqrt{3}}{5000000000000000000000000000}$$

$$\mathbf{ratio}:=\frac{\mathbf{curvtr}}{\mathbf{tr}} \blacktriangleright \frac{124000229512677\cdot\sqrt{41333409837559}}{7812500000000000000}+\frac{999770487323\cdot\sqrt{5998622923938}}{93750000000000000000}-\frac{124753342892097466206711118987}{1250000000000000000000000000}$$

Other remark : the reasoning conducted by hand could have been this one (simulated in a Note page)

"By hand" method assisted by the CAS

$$\mathbf{yn}^2+4\cdot(\mathbf{xn}-1) \blacktriangleright \frac{3\cdot xm^2}{4}+xm\cdot\left(\frac{ym\cdot\sqrt{3}}{2}-2\right)+\frac{ym^2}{4}+2\cdot ym\cdot\sqrt{3}$$

$$\text{expand}\left(\mathbf{yn}^2+4\cdot(\mathbf{xn}-1),\mathbf{yn}\right) \blacktriangleright \frac{3\cdot xm^2}{4}+\frac{xm\cdot ym\cdot\sqrt{3}}{2}-2\cdot xm+\frac{ym^2}{4}+2\cdot ym\cdot\sqrt{3}$$

$$\text{expand}\left(4\cdot\left(\mathbf{yn}^2+4\cdot(\mathbf{xn}-1)\right),\mathbf{yn}\right) \blacktriangleright 3\cdot xm^2+2\cdot xm\cdot ym\cdot\sqrt{3}-8\cdot xm+ym^2+8\cdot ym\cdot\sqrt{3}$$

$$\mathbf{t}(xm):=ym^2+2\cdot\sqrt{3}\,(xm+4)\cdot ym+3\cdot xm^2-8\cdot xm \blacktriangleright \textit{Done}$$

$$\mathbf{delta}(xm):=\left(\sqrt{3}\,(xm+4)\right)^2-3\cdot xm^2+8\cdot xm \blacktriangleright \textit{Done}$$

$$\mathbf{delta}(xm) \blacktriangleright 32\cdot xm+48 \quad \text{solve}\left(\mathbf{delta}(xm)=0,xm\right) \blacktriangleright xm=\frac{-3}{2}$$

Two functions:

$$f(xm):=-\sqrt{3}\,(xm+4)+\sqrt{\text{delta}(\text{xm})} \blacktriangleright \textit{Done}$$

$$\text{and } \mathbf{g}(xm):=-\sqrt{3}\,(xm+4)-\sqrt{\text{delta}(\text{xm})} \blacktriangleright \textit{Done}$$

....

$$\mathbf{f}(xm) \blacktriangleright 4\cdot\sqrt{2\cdot xm+3}-(xm+4)\cdot\sqrt{3}$$

4. An attempt to extand these results in 3D

4.1. Cube, tetrahedron, and spheres

4.1.1. Cube and spheres

In figure 71, we construct a cube and display its area and its volume. We construct the first sphere centered at one vertex of the sphere and having as a radius the length of the side of this cube.

In figure 72, we can see the 8 spheres constructed like the previous one. Then we have constructed the cube which vertices are the points of these spheres belonging to the diagonals of the initial cube. The area and the volume of this cube are displayed.

Finally we have calculated and displayed the ratios between these areas and between these volume to get respectively 4.64… and 10.

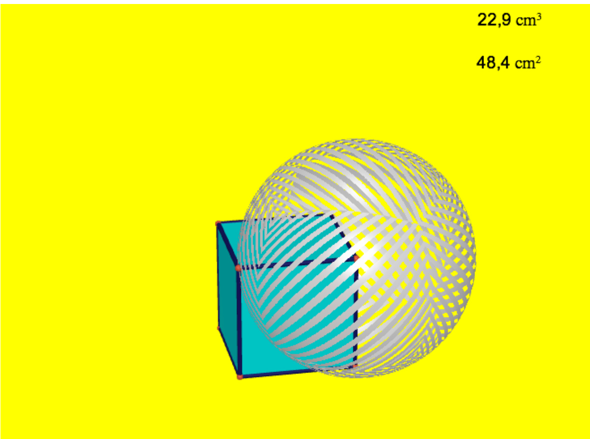


figure 71

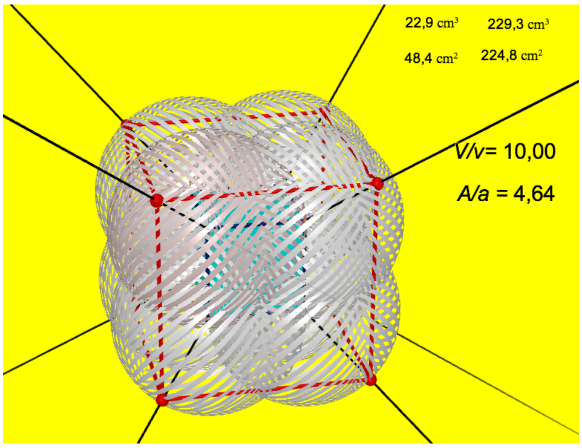


figure 72

4.1.2. Tetrahedron and spheres

In figure 73, we construct a tetrahedron and display its area and its volume. We construct the first sphere centered at one vertex of the tetrahedron and having as a radius the length of the side of this tetrahedron.

In figure 74, we can see the 4 spheres constructed like the previous one. Then we have constructed the tetrahedron which vertices are the points of these spheres belonging to

the altitudes of the initial tetrahedron. The area and the volume of this tetrahedron are displayed.
Finally we have calculated and displayed the ratios between these areas and between these volumes to get respectively 6.93265… and 18.25363.

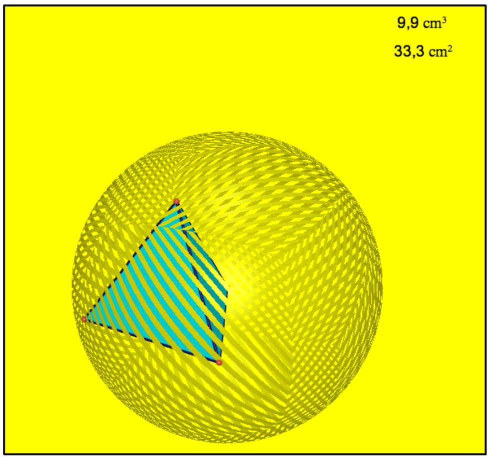


figure 73

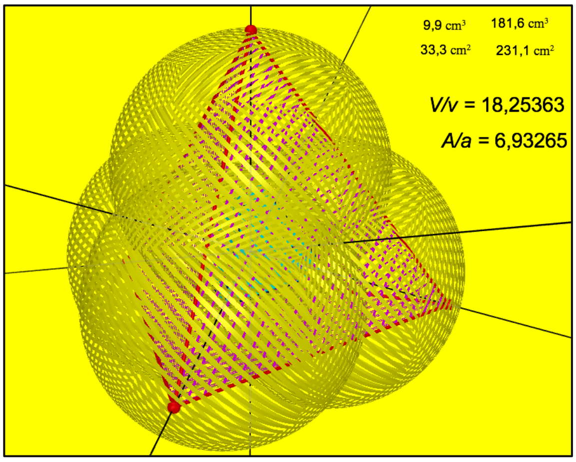


figure 74

4.1.3. Remark : the previous results are experimental results obtained with the help of technology. At this stage of this work, we did not try to find a formal solution.

4.2. Cube, tetrahedron, and parabolas

The idea is to do the same thing in replacing the spheres by paraboloids having the two vertices of every side of the initial cube or tetrahedron as focus and vertex. But, we cannot conduct such experiments with Cabri 3D or the 3D graphing tool of TI N'Spire. We will try to do that in another work with the help of Maple : by now, I am not sure that it is possible to conduct all the stages of this experiment with this software. So we used a trick helping us to conduct such an experiment in using for each paraboloid one of the parabola generating it. We have managed to construct these parabolas in Cabri 3D.

4.2.1. Cube and parabolas

We start with a cube. We construct, as we did in figure 24, on one of its faces, the four parabolas defining the biggest star and the the biggest square which vertices are the most distant points of intersection of these parabolas. Figure 76 shows the construction seen from below and figure 75 from the top.

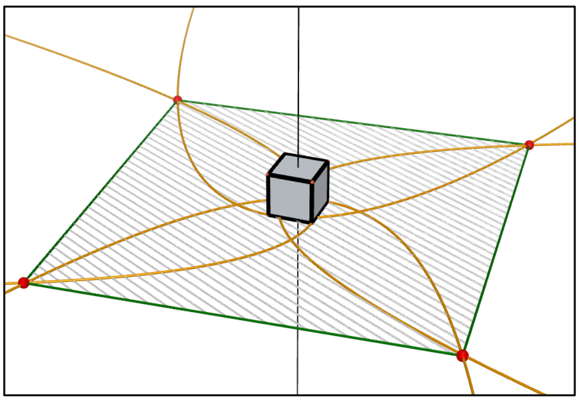


figure 75

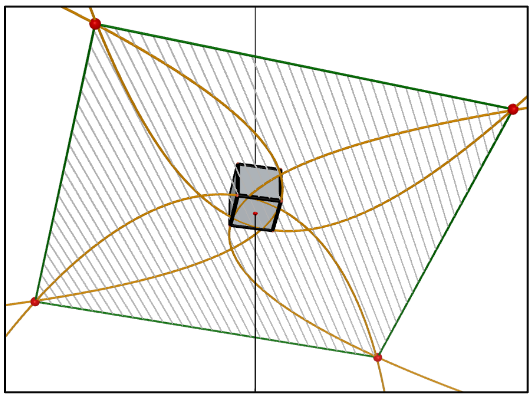


figure 76

Then we repeat these constructions on each face (figure 77) The 24 vertices of the 6 squares define a convex polyhedron shown in figure 78. In the same figure, we have evaluated and displayed the ratio of the areas between this convex polyhedron and the initial square and the ratio of their volume. We have obtained :
Ratio between areas : 66 .98...
Ratio between volumes : 683.88

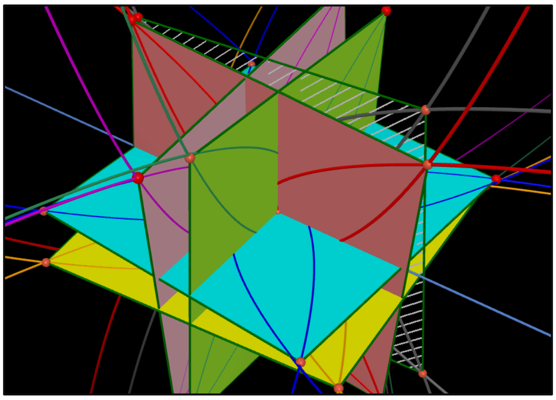


figure 77

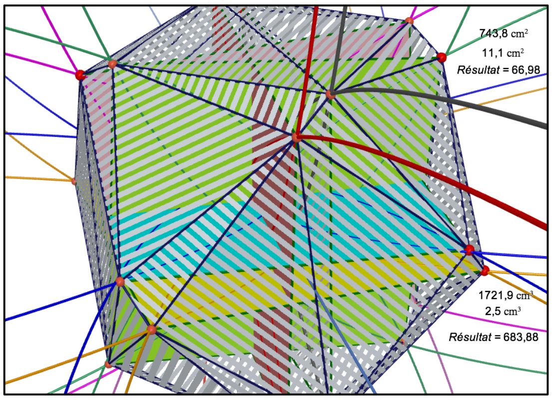


figure 78

4.2.2. Tetrahedron and parabolas
The cube is replaced by a regular tetrahedron. On the plane containing a face, only three parabolas like the parabolas constructed in figure 26 (12 parabolas in figure 79). The 6 squares are replaced by 4 equilateral triangles (figure 80).

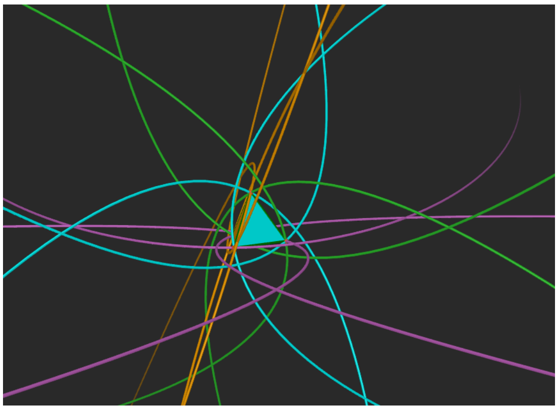


figure 79

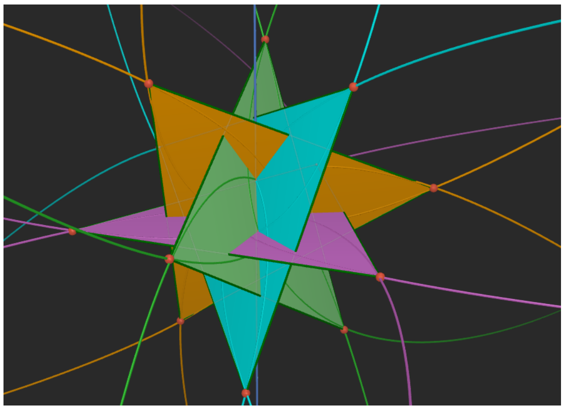


figure 80

Then we construct the convex polyhedron containing this 4 equilateral triangle (or their 12 vertices). The same ratios are evaluated and we have obtained (figure 81) :
Ratio with areas : 65.87...
Ratios with volumes : 870.40...

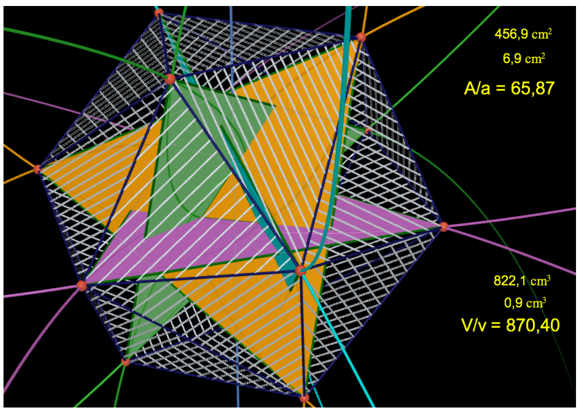


figure 81

4.2.3. Some remarks about the next direction of research

In the previous case, it would be interesting to conduct such experiments for the other platonic solids and to try to find out the shapes of the convex polyhedra we would obtained.

It would be also very interesting to put all the data got during these experiments in a table and treat them statistically to try to find out a relation between them or some invariant.

Another challenge would be to represent in 3D all the paraboloids we have not seen to verify or not if the vertices of our stars and our final convex polyhedra are the vertices of the 3D star defined by these paraboloids. Probably Maple would be the appropriate sofware to manage the construction of all the paraboloids we need and to visualize the 3D stars.

5 Conclusion

This paper shows that starting from very simple constructions within a DGS environment, it is possible to create a very interesting problem which aim is to discover experimentally a relation between some areas of the objects constructed. It shows that changing in a given problem circles by parabolas can enrich the field of our research. It shows that the generalization from 2D to 3D can sometimes be conducted successfuly when using an appropriate software like Cabri 3D. Some proofs had been performed thanks to the CAS of TI N'Spire and especially with the special power of of the Notes pages where all what we have performed can be refreshed instantaneously if we change any of its entries. Another interest of this paper is that each experiment conducted with DGS or CAS can be adapted for students in order for them to investigate successfully problems they could not investigate with paper and pencil. Such experiments and the way of conducting them can open to them a window on what is really a math research work and what are the tools (mathematical and technological ones) used during it.

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[3] Playlist of the YouTube channel « jjdahan » untitled « PRESENTATION JJ DAHAN T3 INTL CONF ORLANDO 2016 » https://www.youtube.com/watch?v=xUtOsy7cLBQ&list=PLOIs4xavv0zEZOpy_ODR0jrU-vX3ycMAJ

Software :

Cabri 2 Plus and *Cabri 3D* by Cabrilog at http://www.cabri.com

TI-Nspire by Texas Instruments at http://education.ti.com/en/us/home