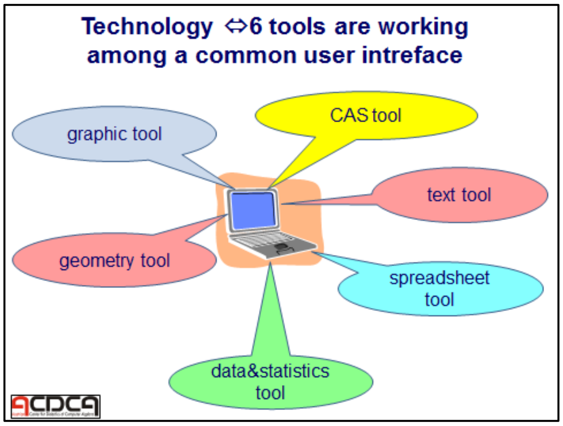
Helmut Heugl

The change from a calculation oriented to a problem solving oriented mathematics education supported/caused by technology

In the Middle Ages there existed an honourable guild, the guild of "calculating masters". They died out when people became able to calculate themselves. Will the math teachers also die out in the age of technology?



What is the role of "technology" in the class-room?

Modern educational Technology implies following tools working among a common user interface.

In Austria we overslept the phase of graphic calculators but on the other side we started very early using CAS in the classroom at the beginning of the nineties.

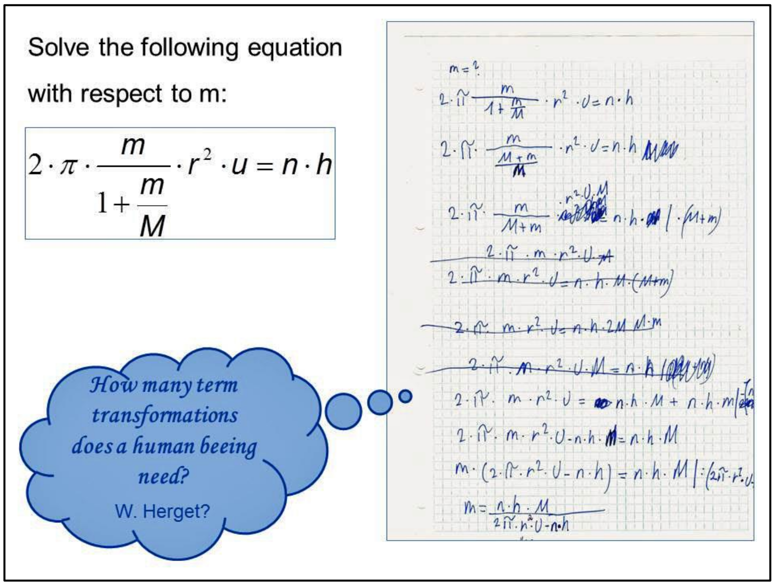
An Austrian journalist required the abolition of mathematics education and in place of it more environmental education:

His justification was: *„I have experienced mathematics as a senseless mental gymnastics".*

If he is right it is necessary that something happens.

Examples of the current situation:

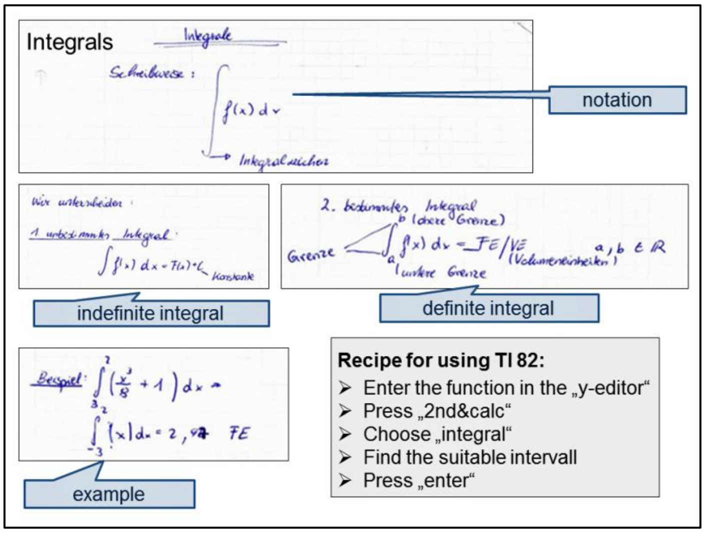
Example 1: Term operations and equations in a technical school (9^{th} grade)



But is the use of technology a possible answer? Not necessarily!

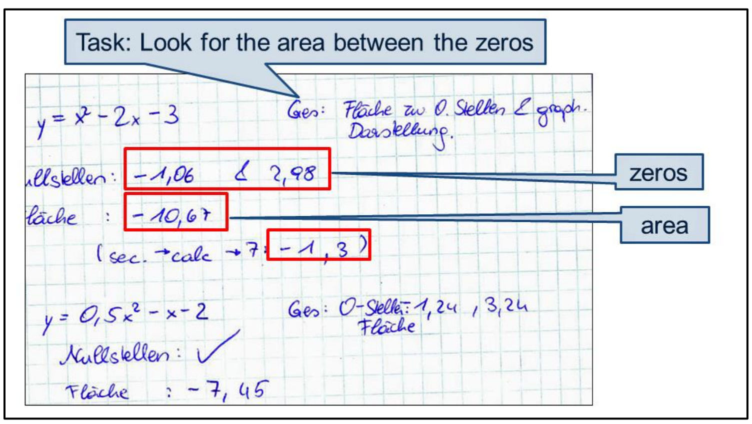
Technology tools could also be dangerous weapons in the hands of some teachers.

Example 2: Teaching Integrals in a vocational school (11th grade). In this class the graphic calculator TI 82 is used.



In the textbooks one can find this definition without any verbal information about this "permanent wave". If there are numbers or variables below and above the integral symbol it is called "definite integral" which are used for calculating areas or volumes.

Afterwards follows the first example associated with a recipe.



The task of the next example is to look for the area between the graph of the function and the x-axes between the zeros. The zeros were found by walking in the trace mode along the graph. So she found -1.09 and 2.98. The real zeros are -1 and 3. By using their recipe they find a solution. The solution of the tool is negative. The teacher interpreted: "Areas below the x-axis are negative."

Even though I criticize that the current mathematics education is too calculation oriented calculation competence is not becoming redundant in the age of technology. But we need a new definition of this competence:

> ***Calculation competence*** *is the ability of a human being to apply a given calculus in a concrete situation purposefully,*

Examples which show the benefit of CAS

Example 3: Complex expressions – different goals

The traditional goal of this example is to simplify the following expression.

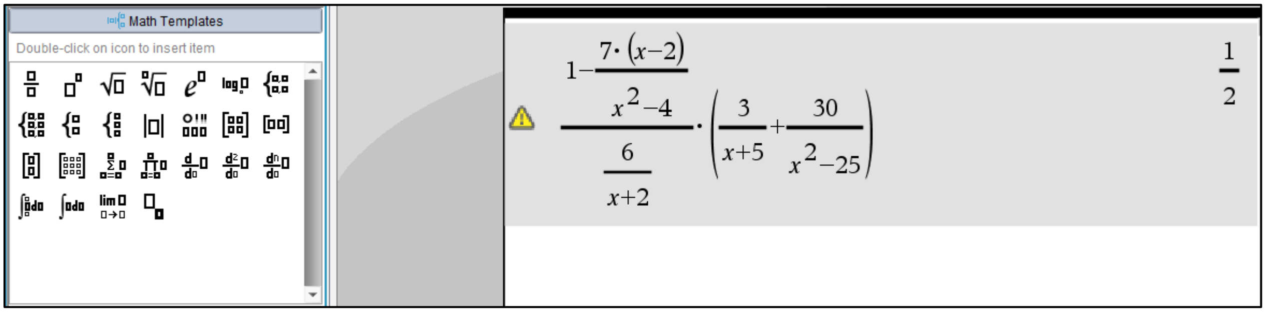
$$\frac{1-\dfrac{7.(x-2)}{x^2-4}}{\dfrac{6}{x+2}}\cdot\left(\frac{3}{x+5}+\frac{30}{x^2-25}\right)$$

A lot of tem operations are necessary. We should ask: Is this competence necessary in the age of CAS?

New goals: The same expression could be used for new goals: Such a goal is to support the important competence of “structure recognition” and the necessary “tool competence”.

Enter the following expression

- by using the math templates of TI Nspire
- by using brackets



Using the Math Templates and pressing “*enter*” shows: The simplified result is a half.

Especially using brackets is not so easy. When I prepared this example I also needed two attempts.

First attempt:

$$\left(1-7\cdot(x-2)/\left(x^2-4\right)/(6/(x+2))\right)\cdot\left(3/(x+5)+30/\left(x^2-25\right)\right)$$

$$\left(1-\frac{\frac{7\cdot(x-2)}{x^2-4}}{\frac{6}{x+2}}\right)\cdot\left(\frac{3}{x+5}+\frac{30}{x^2-25}\right) \qquad \frac{-1}{2\cdot(x-5)}$$

Entering the first version shows: Something is wrong.

Second attempt:

$$\left(\left(1-7\cdot(x-2)/\left(x^2-4\right)\right)/(6/(x+2))\right)\cdot\left(3/(x+5)+30/\left(x^2-25\right)\right)$$

$$\frac{1-\frac{7\cdot(x-2)}{x^2-4}}{\frac{6}{x+2}}\cdot\left(\frac{3}{x+5}+\frac{30}{x^2-25}\right) \qquad \frac{1}{2}$$

The second attempt was successful. The competence of recognizing structures was necessary.

The calculation competence is still important but **new points of view** are necessary:

- Shift of the focal point from operating to modelling and arguing
- Shift of the focal point from doing to planning
- Shift of the focal point from calculating by hand to other algebraic competences like structure recognition, finding expressions, testing a.s.o.
- Less complexity of expressions by calculating by hand
- Necessity of tool competence
- More real life problems
- More space for questioning the mathematical theory

But before we discuss questions like

Why technology?
How technology?
What sort of technology?

We have to think about

Why mathematics? What is the goal of mathematics education?
Not until we have considered the contribution of mathematics to a general education should we think about the role of technology for a better mathematics education.

A possible description of the importance of mathematics is the following

> *While the focus of primary and secondary I education is on the general living environment, in the higher general education learners should experience mathematics as a special path towards worldly wisdom,* ***as spectacles for recognizing and modeling the world around them****. That requires the acquisition of* ***the thinking technology which is significant for doing mathematics*** *and which is the base of a general problem solving competence.*

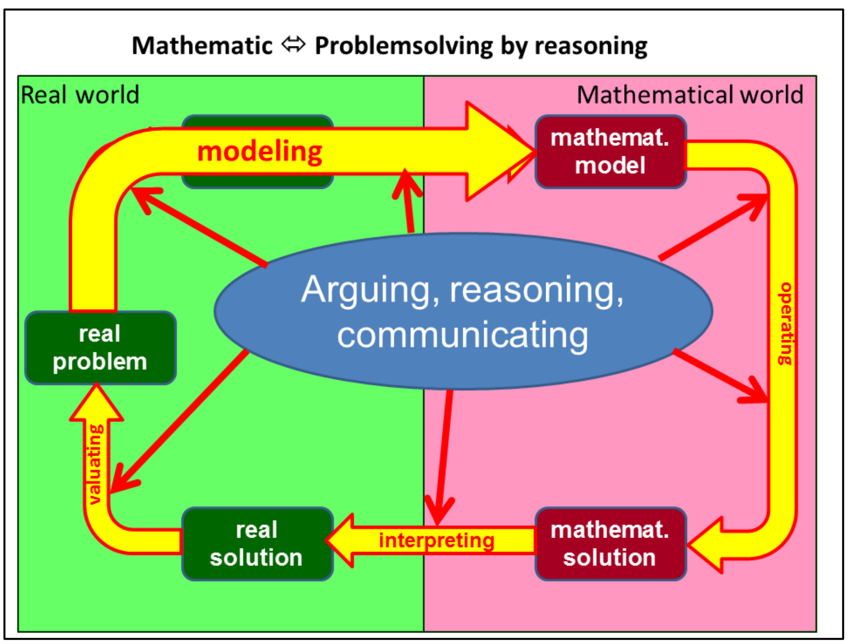
This definition is part of an "educational theory" which we formulated at the beginning of the concept of a new Austrian final exam.

Further aspects formulated by one of the most important mathematicians Bruno Buchberger:

- The goal of mathematics is automation.
- The goal of mathematics is to trivialize mathematics.
- The goal of mathematics is explanation (= making things of high dimension plain = making complicated things simple).
- The process of trivialization is completely non-trivial.
- Think nontrivially once and act trivially infinitely often.

My favorite definition of mathematics is the guideline of this lecture:

> ***Mathematics is the technique of problem solving by reasoning, refined throughout the centuries***



The problem solving process often starts with real problems with data, with verbal information. This information has to be condensed and structured to come to a so called "real model" which can be translated into the language of mathematics.
This part of the process which leads to a mathematical model we call **modeling.**

By **operating** a mathematical solution should be found. **Interpreting** leads to a real solution which has to be evaluated to decide the solution of the problem. **Arguing and reasoning** accompanies the whole process.

Let us look at the first phase, **the phase of modeling** more detailed:

Two aspects should be considered:

Aspect 1: Mathematics is a language and like other languages it has its own grammar, syntax, vocabulary, word order, synonyms, conventions, a.s.o. This language is both a means of communication and an instrument of thought.

My thesis which I will corroborate by the following examples:
Technology supports the translation process from the native language into the language of mathematics

Aspect 2: We have to analyse the development of concepts in the course of the learning process: *General concepts are made cognitively available by prototypical representation.*

Think about the concept table: A little child does not acquire this concept by a precise definition offered by the father but by experiencing concrete prototypes of tables.

My thesis: The computer as a medium of prototypes: The comp*uter offers not only a larger variety of prototypes, but rather, and more importantly, those which would not be available without the computer.*

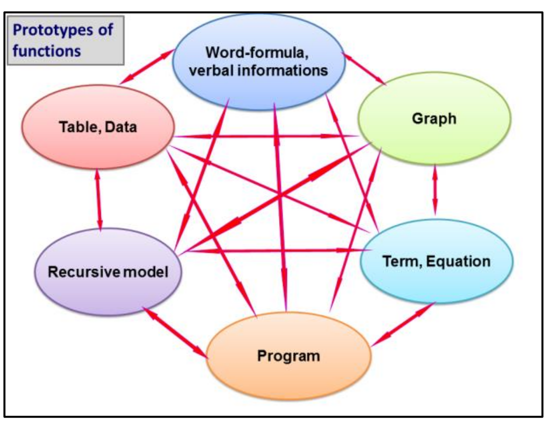
Typical for this thesis is the **concept of functions**. The pupil will find access to this concept through a supply of suitable prototypes which draw the pupil´s attention to the vital characteristic of the concept. In this process the important activity is the establishment of relationship among the individual prototypes. It is in this way that the learner can comprehend that the individual prototype is simply one of many possibilities of appearance of the concept of function. Not until after this process does it make any sense to verbalize or formally define the concept „function".

Observing traditional mathematics education you can find the following **prototypes of the fundamental concept of functions**:

- word formulas
- symbolic prototypes like terms, parametric equations, polar equations
- graphs
- tables

The computer also offers new prototypes e.g.

- recursive models
- programs



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It is not enough to make various prototypes of a general concept available, the establishment of the relationship among the individual prototypes leads the pupil´s attention from the specificity of a singular prototype to the superior general concept.

In traditional mathematics education prototypes mostly are available in a serial way. A typical example is the discussion of curves: One prototype, the term is given. The students have to find the graph by calculating the zeros, the extreme values, the inflection points and they have to determine a table of values.

The main importance of the computer is that **the learner can use several prototypes parallelly**. The given term allows the pupil to draw the graph directly and the table is the result of activating one key of the tool. The real learning process consists of **shuttling between several prototypes** and investigating the influence of changes of one prototype in the others. Therefore we call this didactical concept the **Window Shuttle Method**.

Now let us **walk along the problem solving** circle by looking at concrete examples:

Concerning the modeling phase we can differentiate between two types of problems:

Problems of Type (1)

The model is given e.g. as a function term, a recursive model, a graph a.s.o.
Tasks could be

- Changing the representation mode e.g. given is the term – look for the graph (in traditional mathematics education a "curve discussion"
- Looking for several characteristics of the function (e.g. maxima) by calculating, drawing, developing tables a.s.o.
- Using the model for finding models of additional problems
- Testing the correctness of the mathematical propositions or operations
- Testing the suitability of the model

Problem solving examples of Type (1)

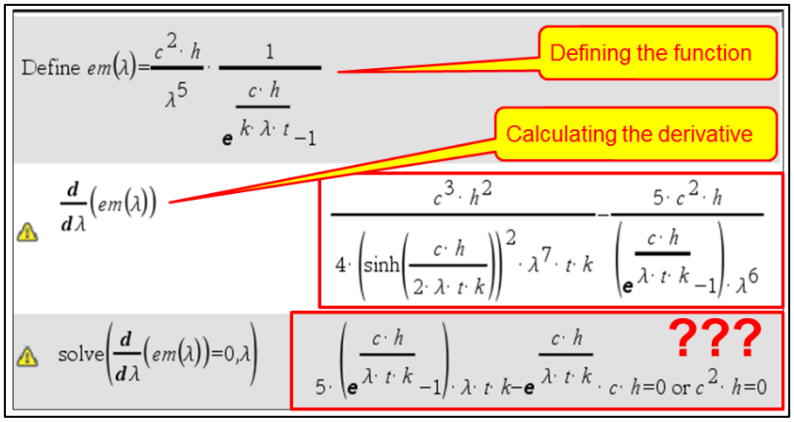
Example 4 Type (1): Given is the term prototype of a function

Problem 1: Planck's radiation formula - surprises by using the strategy of substitution [Dorninger, 1988]

The emission "em" of a black corpus is a function of the wavelength λ.

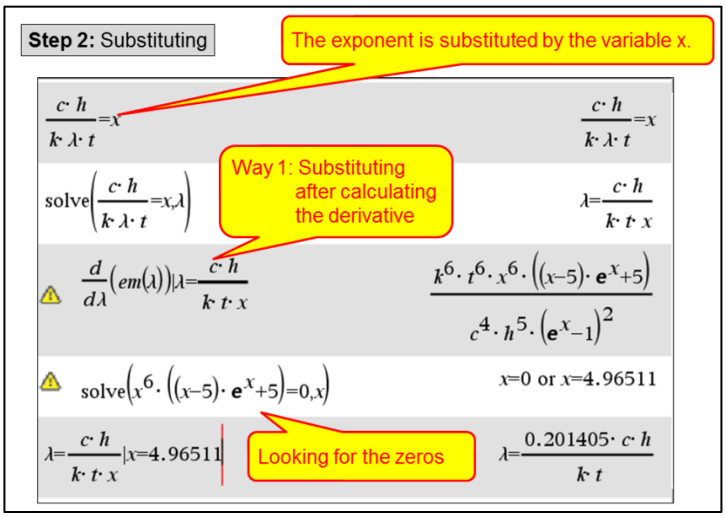
$$em(\lambda) = \frac{c^2 \cdot h}{\lambda^5} \cdot \frac{1}{e^{\frac{c \cdot h}{k \cdot \lambda \cdot t}} - 1} \qquad \text{c,h,k,t are constants } (\neq 0)$$

The task: Determine the maximum value of the function.



Step 1: Looking for the zeros of the derivative.

Solving the equation shows the limits of the CAS-tool and needs the mathematical **competence of substituting** by the students. Again we can see that the learners are planning the tool is doing.



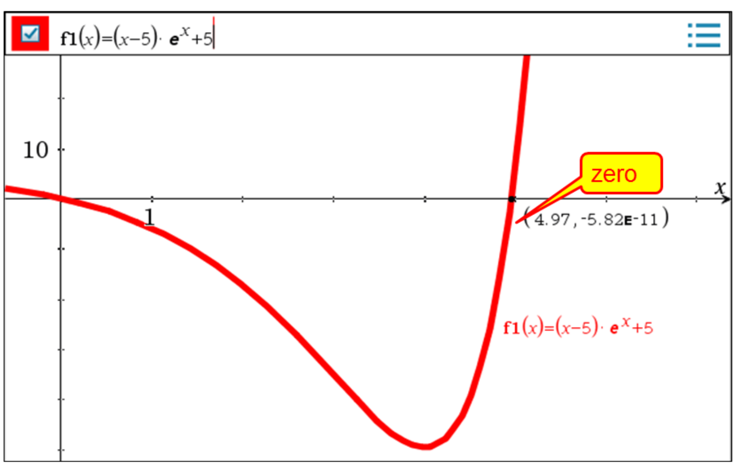
Step 2: Substituting

The exponent is substituted by the variable x.

Two several ways are thinkable:

Way 1: Substituting after calculating the derivative

Students think: When looking for the zeros of the derivative it will do to look for the zeros of the numerator. A reasonable value of x is 4.96511. When substituting for x we get the solution for λ (with respect to the constants c, h, k, t.



Step 3: Another option is to solve the equation $(x-5).e^{x}+5=0$ graphically by looking for the zeros of the function f1.

Way 2: Substituting before calculating the derivative

$$\frac{d}{dx}\left(em(\lambda)|\lambda=\frac{c\cdot h}{k\cdot t\cdot x}\right) \qquad \boxed{\frac{-k^5\cdot x^4\cdot\left((x-5)\cdot e^x+5\right)\cdot t^5}{c^3\cdot h^4\cdot\left(e^x-1\right)^2}}$$

$$\text{solve}\left(x^4\cdot\left((x-5)\cdot e^x+5\right)=0,x\right) \qquad x=0 \text{ or } x=4.96511$$

$$\lambda=\frac{c\cdot h}{k\cdot t\cdot x}|x=4.96511 \qquad \lambda=\frac{0.201405\cdot c\cdot h}{k\cdot t}$$

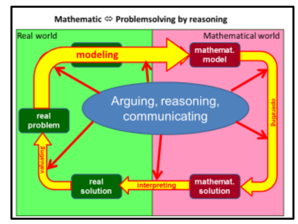
Way 2: Substituting before calculating the derivative

The numerical solution is the same.

But let us look again at the problem solving circle

Arguing and reasoning accompanies the whole process

When reflecting about the mathematical results which the tool offers an **exactifying phase** would be necessary.



Possible questions are:

Question 1: x = 0 is a solution of the following equation:

$$\text{solve}\left(x^6\cdot\left((x-5)\cdot e^x+5\right)=0,x\right) \qquad x=0 \text{ or } x=4.96511$$

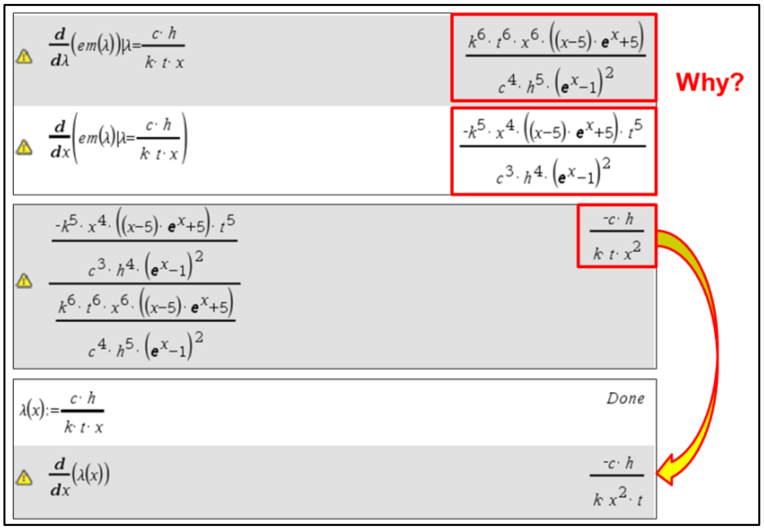
Is it really a zero of the derivative?

$$\frac{d}{d\lambda}\left(em(\lambda)\right)|\lambda=\frac{c\cdot h}{k\cdot t\cdot x} \qquad \frac{k^6\cdot t^6\cdot x^6\cdot\left((x-5)\cdot e^x+5\right)}{c^4\cdot h^5\cdot\boxed{\left(e^x-1\right)^2}}$$

Is it enough to look at the numerator when looking for the zeros of the derivative?

Is x = 0 also a solution?

At first we have to look for the domain of the function by looking for the zeros of the denominator and we can see that for x = 0 the term (e^x – 1) is 0.



The comparison of the derivatives with respect to x of way 1 and way 2 shows: They are not equal? How could we explain the disparity?

A commonly used testing strategy is to build the quotient or the difference of the two expressions:

What is the meaning of this quotient?

If we remember that λ is a function of x we can calculate the inner derivative of the function λ(x) and we see that the quotient of the derivatives of the two ways is the inner derivative.

Recursive models for irrational numbers

If you ask a student, “what is $\sqrt{2}$?”, a supposable answer would be: “*It is a number which squared is equal 2.*” But what’s that supposed to mean? In fact it is the confession not to be able to solve the equation $x^2 = 2$.

A typical quality of mathematicians is that they are vain. If they are not able to solve a problem they invent new mathematical objects.

A good definition of irrational numbers which implies also the idea of the extension of the rational to the real numbers is the following:

> ***The step to irrational numbers takes place by declaring the possibility of an arbitrary approach as a number***

By using recursive models technology enables the students to experience this definition.

Example 5 Type (1): Given is a recursive model of a function

Problem: Recursive models for the approximation of irrational numbers, especially $\sqrt[k]{a}$

[Schweiger, F.]

Given are two recursive models:

Model 1: $x_{n+1} = \frac{1}{2} \cdot \left(x_n + \frac{a}{x_n^{\,k-1}} \right)$ and Model 2: $x_{n+1} = \frac{1}{k} \cdot \left((k-1) \cdot x_n + \frac{a}{x_n^{\,k-1}} \right)$

- **Task:** Investigate the convergence of two models for the approximation of $\sqrt[k]{a}$

The learning process proceeds in two phases:

- **Phase 1** (experimental phase): Draw the graphs of the sequence in the "time mode" and the "web mode". Is a convergence observable?
- **Phase 2** (exactifying phase): Calculate the fixed points of the sequences and investigate the character of the fixed point by using the fixed point theorem.

Phase 1 (experimental phase): Investigating the graphs, coming to suppositions

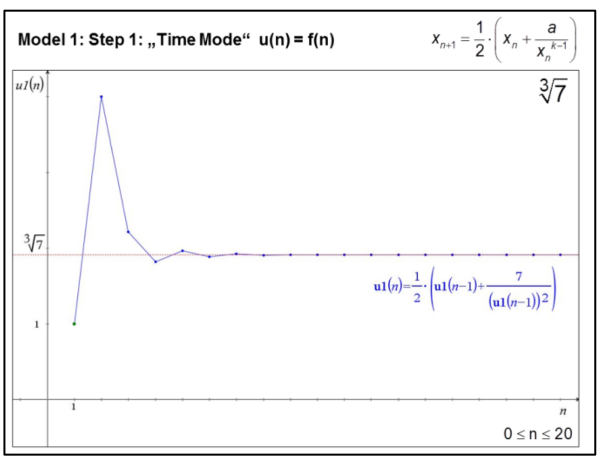
Students can use two representation modes of the graph:

- Step 1: They use the „Time Mode" u(n) = f(n)
- Step 2: They use the „Web Mode" u(n) = g(u(n-1))

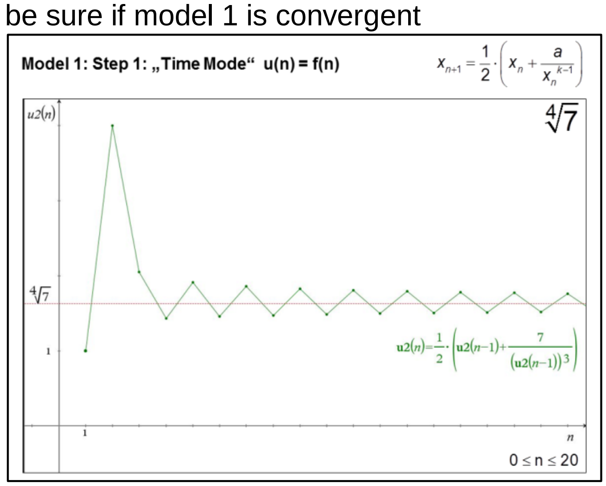
For the experimental phase I have chosen: a = 7 and k = 3, 4, 5

Model 1:

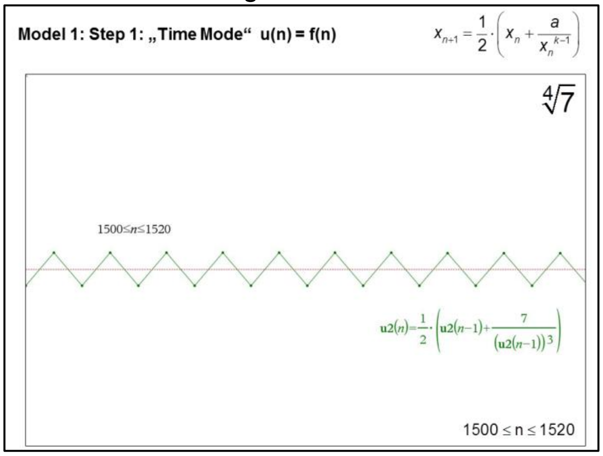
k = 3: Model 1 seems to be convergent



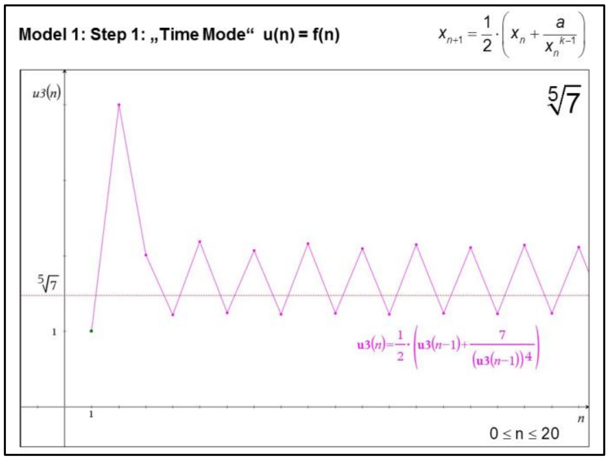
k = 4: Observing the interval [1,20] we cannot be sure if model 1 is convergent



k = 4: But by changing the window variables we look at other regions.

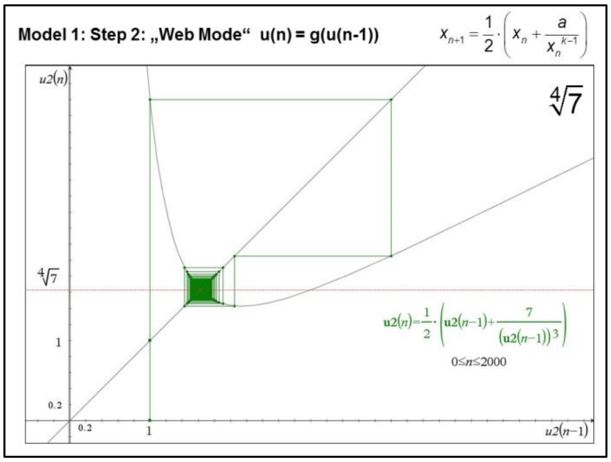


k = 5: It is improbable that model 1 converges

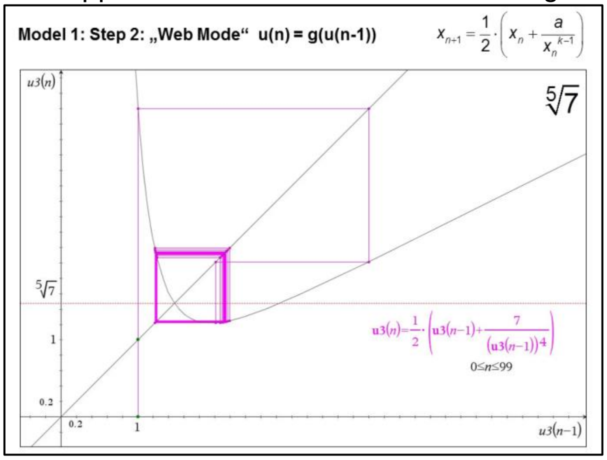


Let us look at the **"web-mode"**:

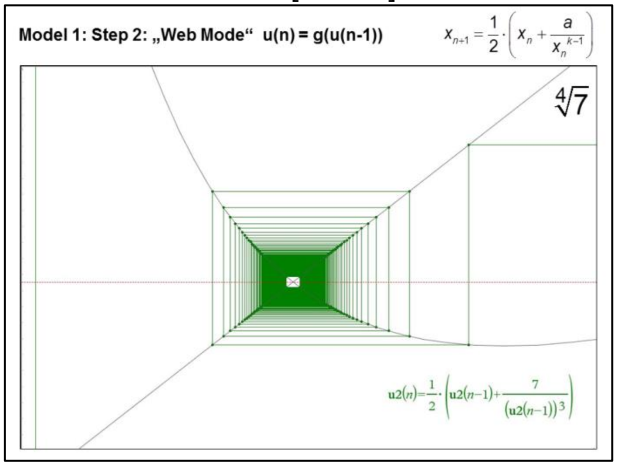
k = 4: Model 1 seems to be convergent



k = 5: The graph in the web mode strengthens the supposition that model 1 is not convergent.

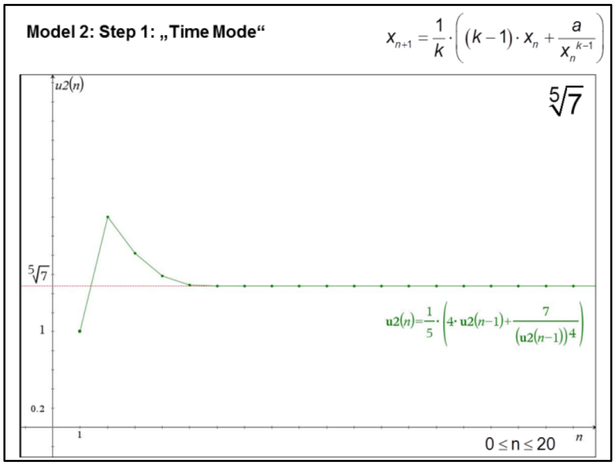


k = 4: When being critical we we zoom in and now we see it will probably not be convergent. But we have still no sureness. We have only visualized the interval [0,2000].

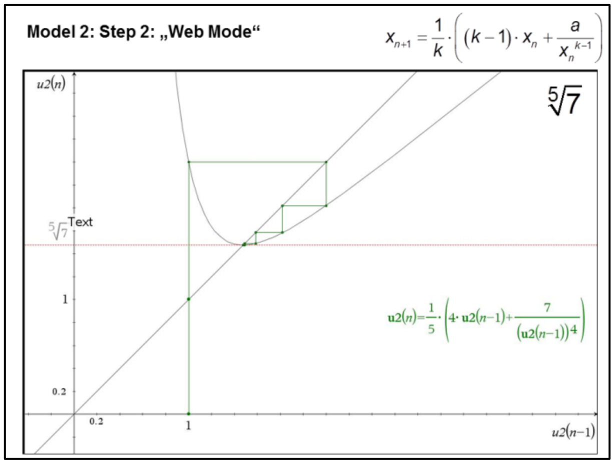


Model 2:

k = 5: Model 2 seems to be convergent also for k=5.



k = 5: The web graph strengthens the supposition.



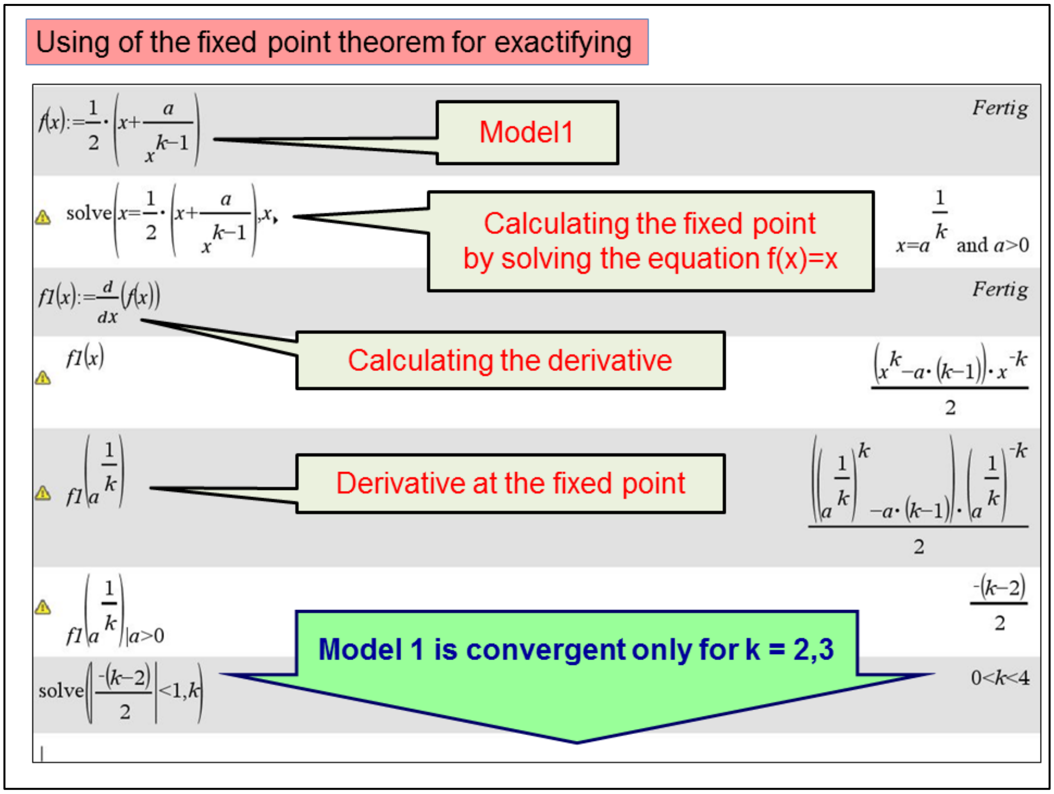
Phase 2 (exactifying phase): Proving the convergence of the given sequences

I think this example clearly shows the possibilities but also the limits of the use of technology in the experimental learning phase. This learning phase often leads the learners to assumption and then in an exactifying phase the power of CAS is necessary to get certainty.

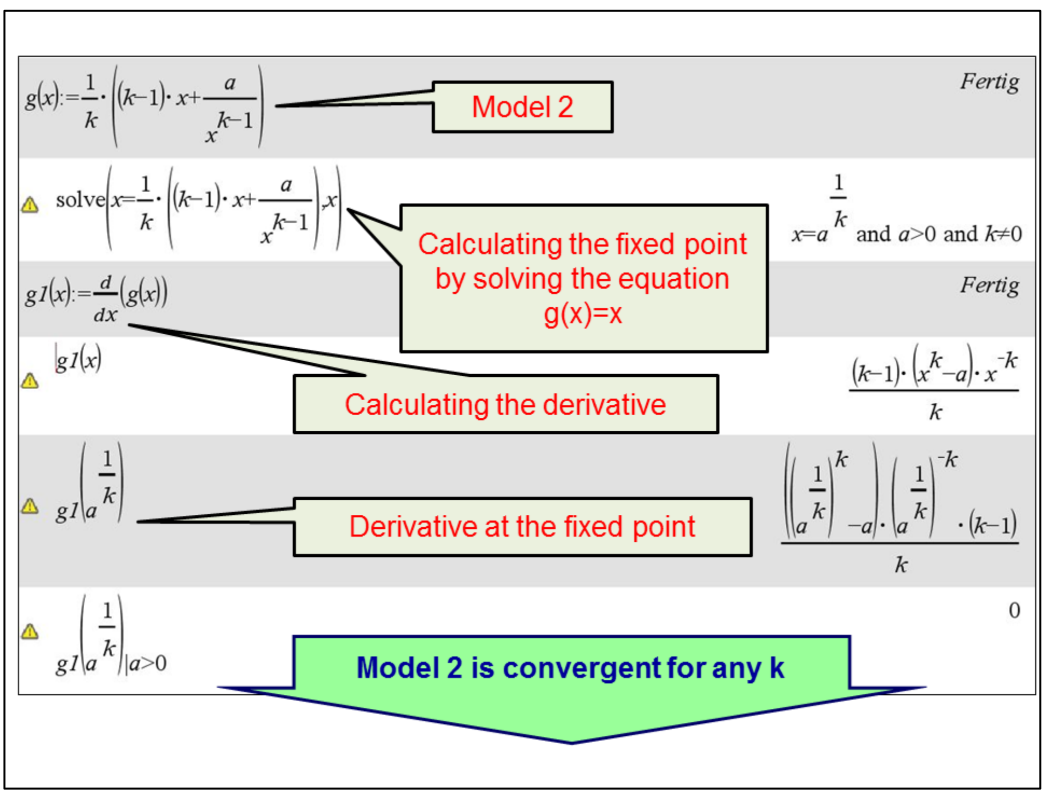
At first we need **some theoretical prerequisites**:

- **A fixed point x*** of a function f is an element of the function's domain that is mapped to itself, that is f(x*) = x*.
- A fixed point is **an attractive fixed point x*** of a sequence given by a difference equation $x_n = f(x_{n-1})$, if f converges to x*, that is $\lim_{n\to\infty} f(x_n) = x^*$.
- **The fixed point theorem:** A fixed point x* of a difference equation $x_n = f(x_{n-1})$ *(f is continuous and differentiable) is an attractive fixed point, if* $|f'(x^*)| < 1$ *and is distractive, if* $|f'(x^*)| > 1$.

The task of phase 2 is: Calculate the fixed points and investigate the character of the fixed point by using the fixed point theorem.



Without knowing the condition that the radicand a must be positive students will not get suitable results. The CAS tool is critical: When looking at the solution of the equation f(x)=x we will find the condition which the CAS tool expects: a>0.



Possibly some of you have recognized that behind model 2 “Newtons Method” is hidden.

Problems of Type (2):

Given are several data or verbal information which should be used to find a mathematical model.

The model can be found by

- deciding a certain type of functions
- using data for calculating the function term
- translating the given information into the language of mathematics
- Looking for a recursive model which allows the direct translation of the given information

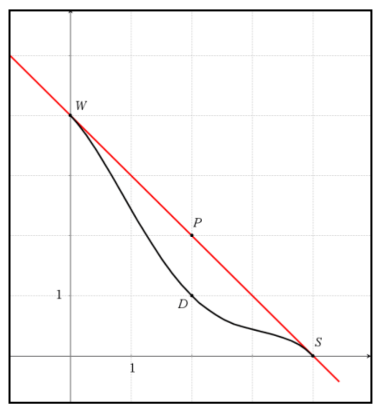
Problem solving examples of Type (2)

Examples 6 Type (2): Translating the given information into the language of mathematics

Problem: A Local Bypass
[An example published as a preparation of the final exam in Austria]

A large village (in the map given with P=(2|2)) is located on a straight road between the villages W=(0|4) and S=(4|0).

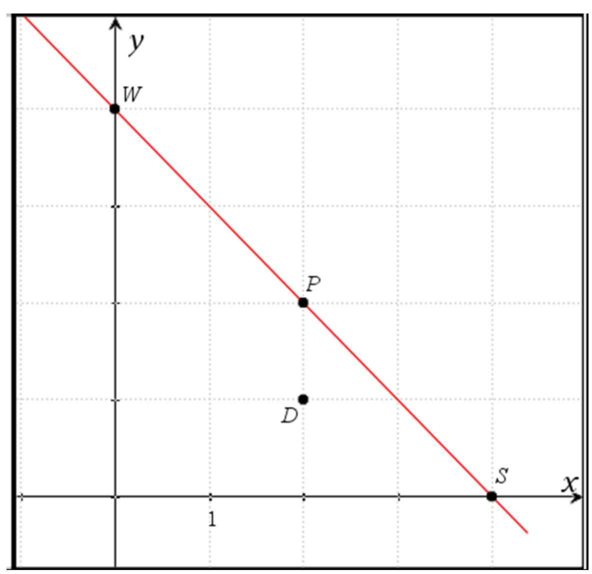
A local bypass should be built which passes the point D=(2|1) and which discharges into the straight road tangentially at W and S. The coordinates are given in miles.



Task:
Find a polynomial function of the local bypass by using all given information and draw the graph of the local bypass and the straight road (x:y = 1:1)

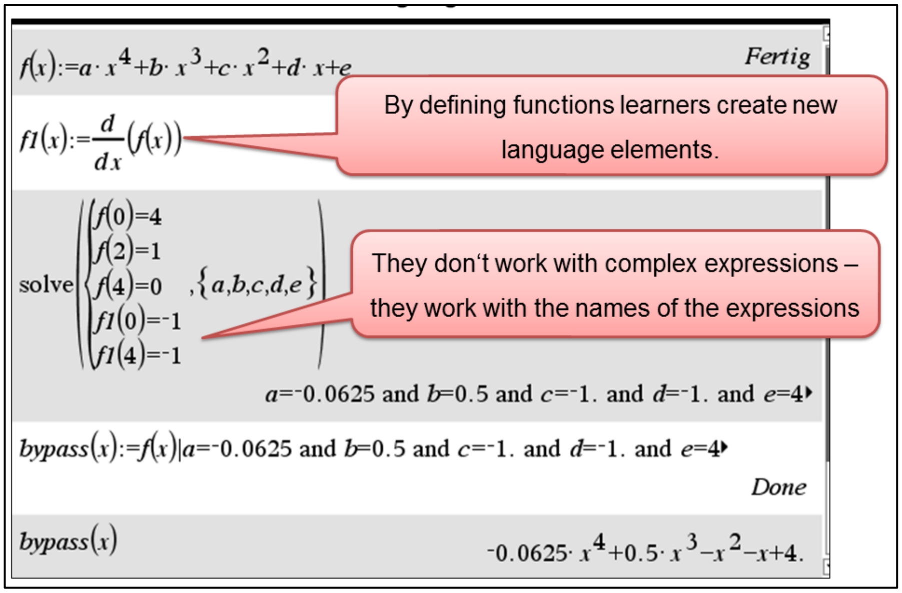
Step 1:

Using the information for a graphic representation of the given problem.



Step 2:

„Mathematizing“: Verbally formulated conditions are translated into the language of mathematics

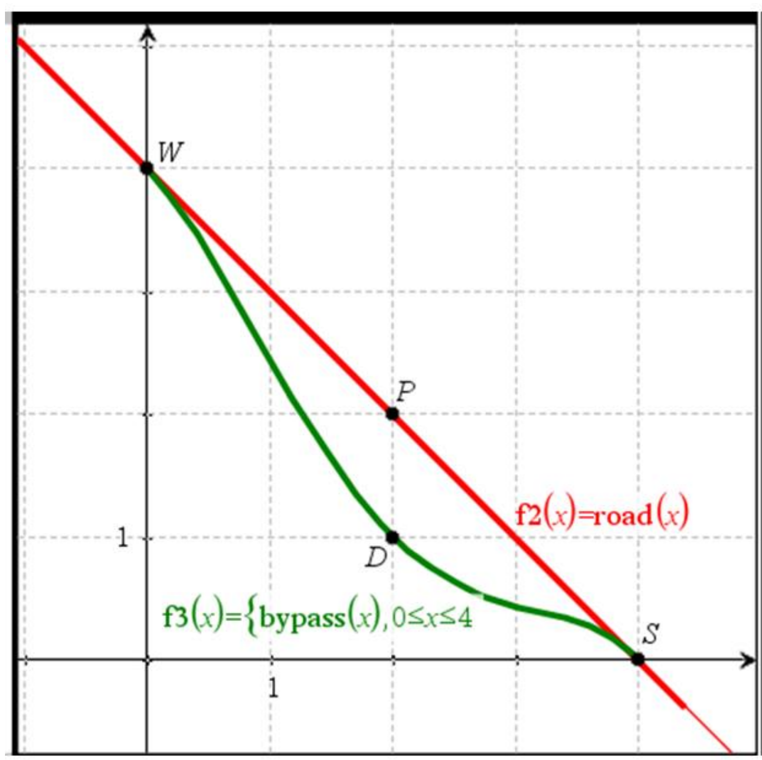


Students have to use the 5 conditions to find a polynomial function with grade n.

This example shows that a CAS tool is not only a calculation tool which supports cognition – it changes cognition: Students don’t work with complex expressions, they create new language elements and work with the names of the expressions. This fact allows a direct translation of verbal information into the language of mathematics.

Step 3:

The next step is to draw the graph of the bypass and the road by defining a piece-wise function.



Example 7 Type (2): Given are data - using the regression functions offered by the tool

Problem: A computer virus [Dangl, 2007]

Within two hours nearly all PCs in a company with 4500 networked PC-working stations are attacked by a virus. After a complete cleanup the network administrator can reconstruct the time flow:

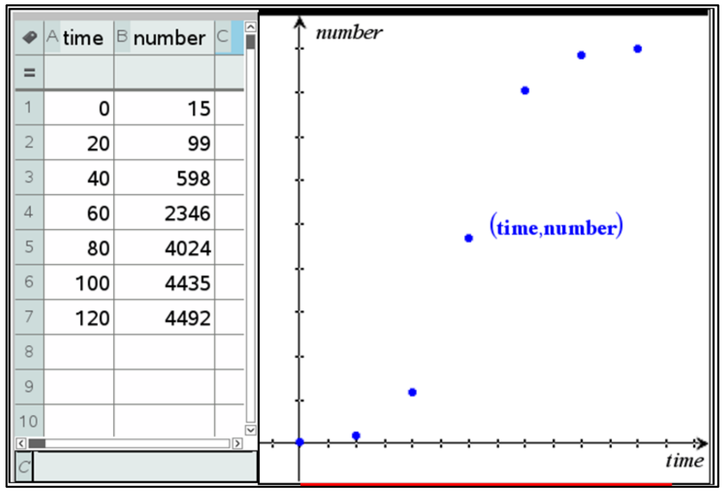
Time (minutes)	0	20	40	60	80	100	120
Number of the infected PCs	15	99	598	2346	4024	4435	4492

Task: What function could describe the growth process best?

Step 1:

Entering data in the Lists&Spreadsheet tool and drawing the graph.

Coming to assumptions concerning a suitable type of functions. A lot of competences about characteristics of several functions are necessary



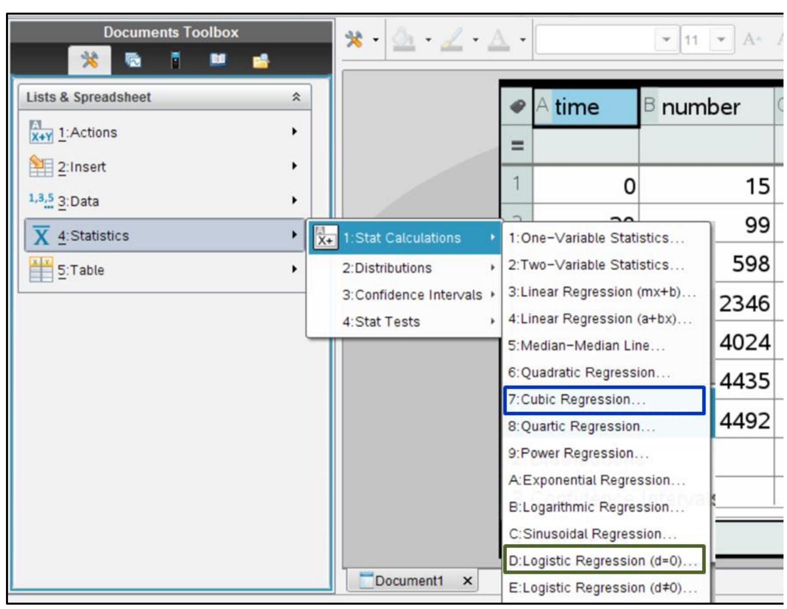
Following assumptions are thinkable:

1st assumption: suitable is a polynomial function

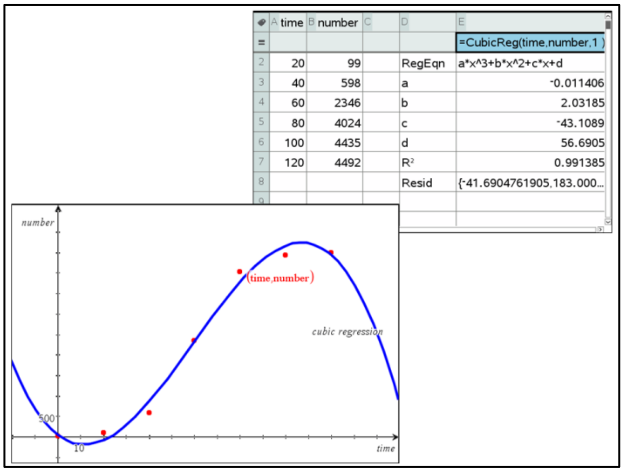
2nd assumption: a possible model is a logistic growth

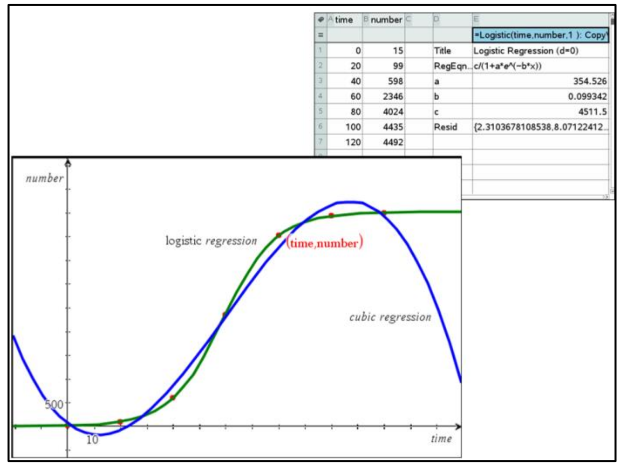
Step 2:

Using the „Lists&Spreadsheet“ application to find the appropriate regression function



1st assumption: A polynomial function 2nd assumption: A logistic growth





For the decision students could use the **coefficient of determination**, denoted $\mathbf{R^2}$ or $\mathbf{r^2}$ and pronounced **R squared. It is** a number that indicates the proportion of the variance in the dependent variable that is predictable from the independent variable.

Example 8 Type (2): Using a recursive model which allows the direct translation of the verbal information

Problem: Applying a loan for buying a house

For buying a hose the bank offers a special sort of loans. At present the rate of interest r is 3.5%. Depending on the index „Euribor" this rate at most can rise to 6%.

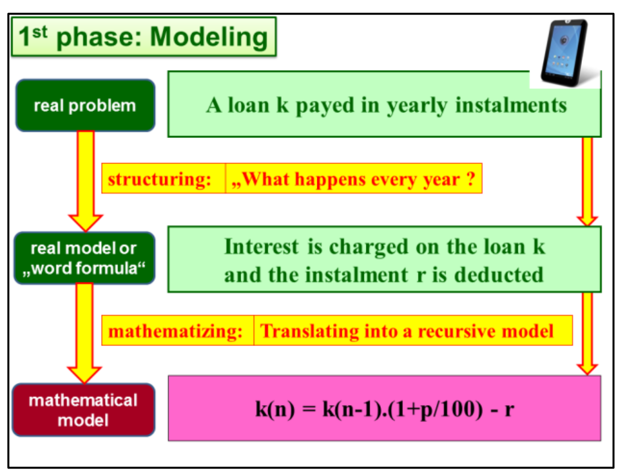
I needed a loan k of € 140 000 and I wanted to repay the loan with annual instalments in 30 years.

a) How much amounts the yearly instalment r paid at the end of the year?

b) How much amounts the yearly instalment if the rate of interest rises to 6% ?

In traditional mathematics education such problems could be solved for the first time in 10th grade, because the students need geometric series and calculating skills with logarithms. The computer offers a new model, the recursive model. From that pupils now work such problems in 7th grade only need fundamental competence in percentage calculation.

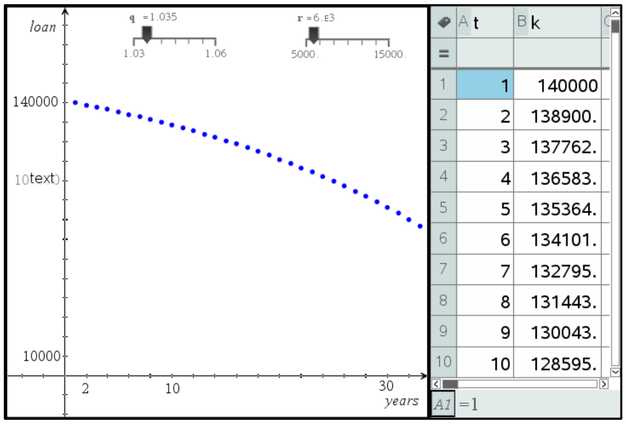
Let us look more detailed on the translation process into the language of mathematics.

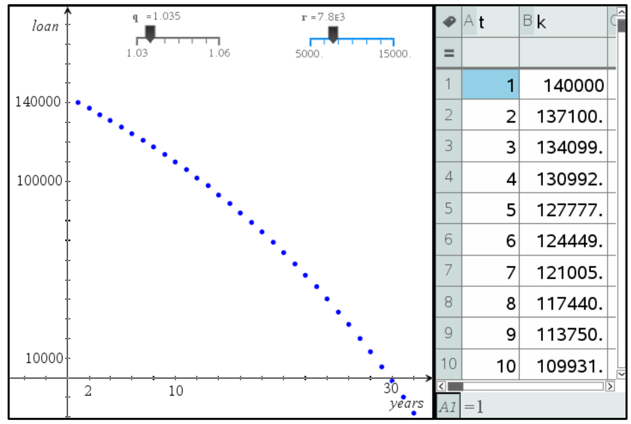


Guidance for the solution with the tool: Define 2 slider bars for the annual instalment r and für the accumulation factor q = (1+p/100) and try to solve the problem graphically.

a) Starting with p=3.5% and r=6300: The graph shows: The yearly instalment must be higher.

Using the slider bar for r shows: The necessary instalment is € 7800.

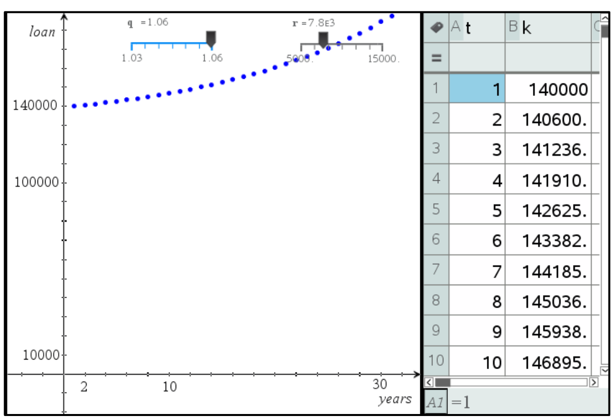


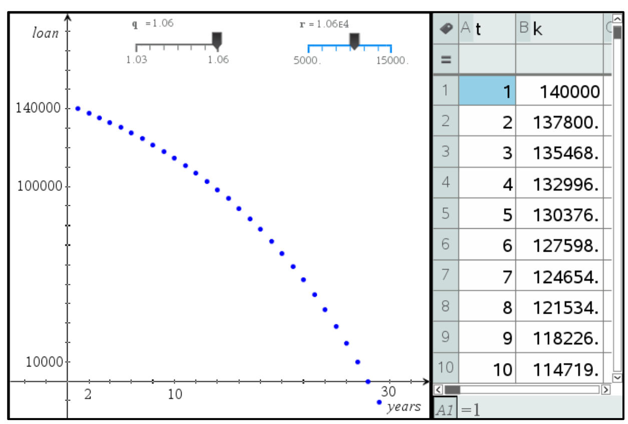


b) But what happens if the rate of interest rises to 6%?

If we would not adapt the yearly instalment, after 30 years the loan will be higher than at the beginning although we have paid € 7800 for 30 years.

The rate of interest of 6% causes a yearly instalment of € 10600!





The last topic of my lecture:

Results of Austrian studies in connection with standard tests [Svecnik, 2009]

Standards are descriptions of fundamental mathematical competences which are expected from all students at the end of secondary level II. In Austria the final exam takes place in the 12th grade.

We tested about 7000 students of the 11th grade these are 33% of the Austrian population in the 11th grade. It was an online test with no use of technology. Accompanying studies investigated the influence of the learning strategies and also of the used technology in the interviewed classes.

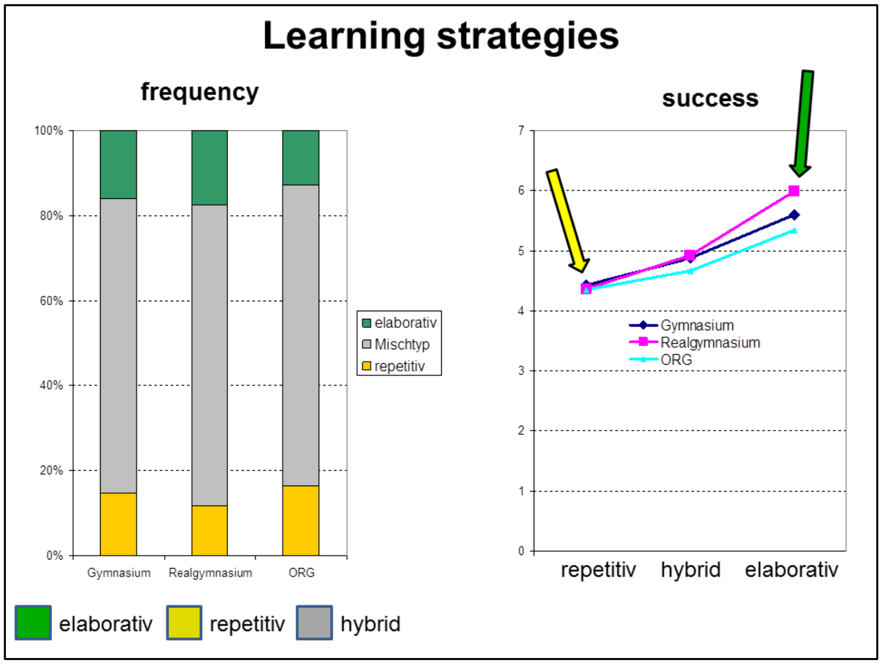
We used a rough classification of individual learning strategies in elaborative and reproductive learners knowing that mostly we find hybrid learning types.

Elaborative learners prefer discovery, exploratory learning.

Reproductive learners limit themselves to repeat the contents and strategies which the teacher offers. Such a strategy is especially successful in a calculation oriented mathematics education. We name this strategy “algorithmic obedience”.

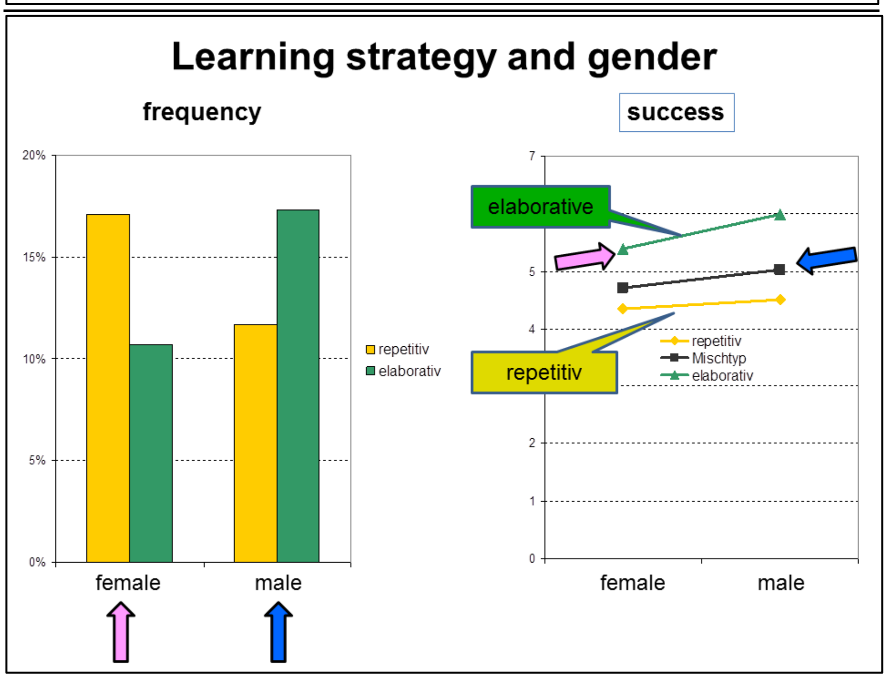
Result 1: The success in function of the learning strategy:

The elaborative learners are significantly better than the reproductive learners.



Result 2: The gender gap:

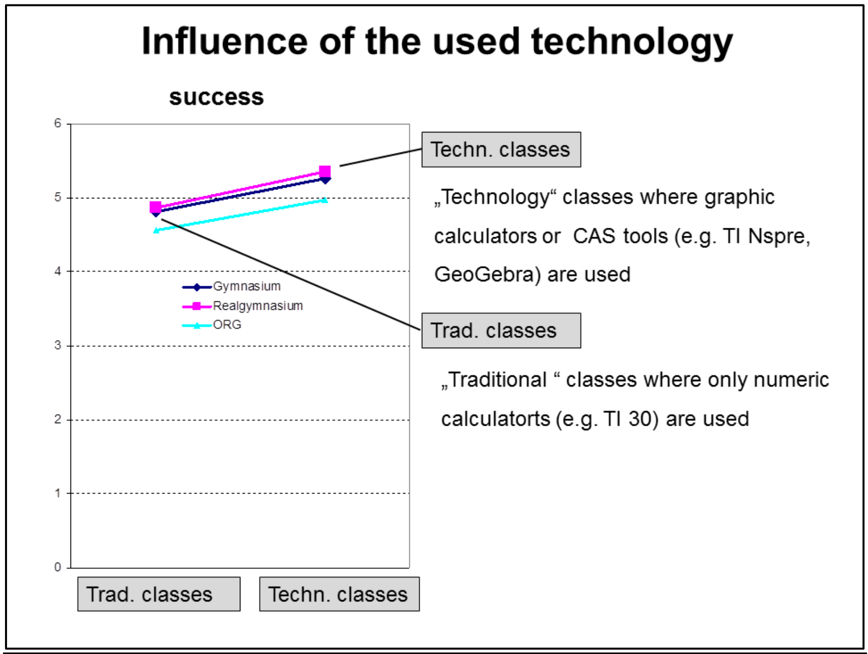
Female learners prefer reproductive learning strategies male learners rather elaborative strategies. Important is that the elaborative learning girls are significantly better than the reproductive learning boys.



The most important result concerning the topic of my lecture:

Result 3: The influence of the used technology

Regular use of technology in all learning situations that is in the classroom, at home and in the exam situation leads to significant better results although technology was not necessary for this test.



Conclusion:

In a calculation oriented math education technology is not necessary - students should calculate „by hand“.	**In a problem oriented math education technology is indispensable**