TIME 2016

# Analysis of non-analytical smooth functions using CAS


Jeanett López García

Jorge Javier Jiménez Zamudio

Ma. Eugenia Canut Díaz Velarde

UNAM-México

# The context

*In mathematics, one of the most important themes which are taught in Bachelor Science is about Taylor expansion for some kind of functions.*

*Besides, if we wanted to solve a differential equation using power series, variable coefficients had to be analytic functions, at a certain point.*

# A mathematical point of view

Washington National Research Council (NRC) (1981) has established that mathematics instruction should focus on developing what Schoenfeld called **"a mathematical point of view"**, i.e.: which includes ability to understand, analyze and visualize structures, and structural relationships in order to understand the whole.

***But which is the best way for succeeding in the goal?***

# Two points of view

**One of them**: Blanchard et al., (1999) (i) the available technology, (ii) the software development and (iii) the easiness of working these at Universities, teaching in the traditional way is no longer appropriate.

**The second:** Hong et al., (2005), have shown there is a problem when students come to rely on CAS since it can then undermine their learning, preventing them from learning important concepts and procedures.

# Even more

*Kells (1978) says, referring at differential equations, that it must give enough theory to provide a thorough understanding and facilitate the acuteness of reasoning.*

*We assume that if we change the issue, differential equations to other mathematical object, like non-analytical functions, must be the same.*

# How can we explain what a non-analytical smooth function is?

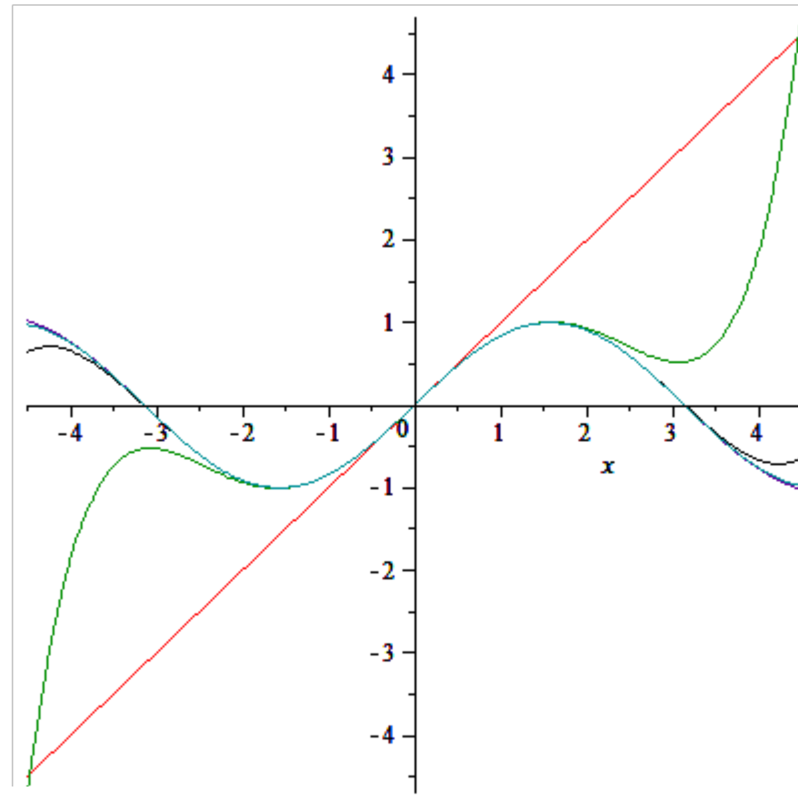
We know that a function f is real analytic on an open set D in the real line if for any $x_0$ in D one can write

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots$$

in which the coefficients $a_0$, $a_1$, ... are real numbers and the series is convergent to f(x) for x in a neighborhood of $x_0$.

# Osculating polynomial

A classic example is




$$f0 := \sin(x)$$

$$f11 := x + \mathrm{O}(x^2)$$

$$f22 := x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \mathrm{O}(x^6)$$

$$f33 := x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 + \mathrm{O}(x^{10})$$

# What is the best strategy for teaching?

## How much technology?

## vs.

## How much theory?

What can I lose or gain if we assume the first position, or the second?

# Non-analytic functions are match with concepts hard to learn

We cannot explain why, but learning power series is one of the most difficult topics to understand in Bachelor students.

There are concepts such as: infinity, convergence, smoothness, continuity, infinitely differentiable, osculating polynomial, and so on which are hard to explain, specially because they do not depend on mathematical operations that students can make with a calculator.

# An example that the device cannot answer

Is

$$\begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & otherwise \end{cases}$$

an analytical function?

CAS cannot answer if the function is analytic or not.

# What happen if I ask Taylor in MAPLE ?

Is it an analytical function?

$$f0 := piecewise\left(x \neq 0, \exp\left(-\frac{1}{x^2}\right)\right) \Rightarrow \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & otherwise \end{cases}$$

$$taylor(f0, x = 0, 4) \Rightarrow$$

Error, (in series/exp)

unable to compute series

# Is it infinitely differentiable?

$$f0 := \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f2 := \begin{cases} 0 & x = 0 \\ -\frac{2\,e^{-\frac{1}{x^2}}\left(3\,x^2 - 2\right)}{x^6} & \text{otherwise} \end{cases}$$

$$f1 := \begin{cases} 0 & x = 0 \\ \frac{2\,e^{-\frac{1}{x^2}}}{x^3} & \text{otherwise} \end{cases}$$

$$f3 := \begin{cases} 0 & x = 0 \\ \frac{4\,e^{-\frac{1}{x^2}}\left(-9\,x^2 + 2 + 6\,x^4\right)}{x^9} & \text{otherwise} \end{cases}$$

# What are we losing?

Using and evaluating the definition in $x_0 = 0$

$$f'(x) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \Rightarrow f'(x) = \lim_{x \to 0} \frac{e^{-\left(\frac{1}{x^2}\right)}}{x} = \frac{0}{0}$$

With l'Hôspital's Rule and MAPLE

$$\lim_{x \to 0} \frac{e^{-\left(\frac{1}{x^2}\right)}}{x} = 0$$

But one more time, if we try to calculate the limit without CAS we have a problem.

# Again, we need mathematics

Fulks (1986) suggests making a substituting $x = \frac{1}{t}$

$$\lim_{x\to 0} \frac{e^{-\left(\frac{1}{x^2}\right)}}{x} \Rightarrow \lim_{x\to\infty}\left(te^{-t^2}\right)$$

With l'Hôspital's Rule and MAPLE

$$\lim_{x\to 0} \frac{e^{-\left(\frac{1}{x^2}\right)}}{x} = 0$$

# The function is infinitely differentiable

If the process is repeated we obtain

$$f''(x) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} \Rightarrow f''(x) = \lim_{x \to 0} \frac{e^{-\left(\frac{1}{x^2}\right)}}{x^2}$$

$$\lim_{x \to 0} \frac{e^{-\left(\frac{1}{x^2}\right)}}{x^2} \Rightarrow \lim_{x \to \infty} \left( t^2 e^{-t^2} \right) = 0$$

By mathematical induction we can prove that is infinitely differentiable

$$f^{(k)}(x) = \lim_{x \to 0} \frac{e^{-\left(\frac{1}{x^2}\right)}}{x^k} = 0$$

# What are the consequences of the last calculus?

We have seen that all for any x in the vicinity of $x_0$, all the values of the derivatives are zero.

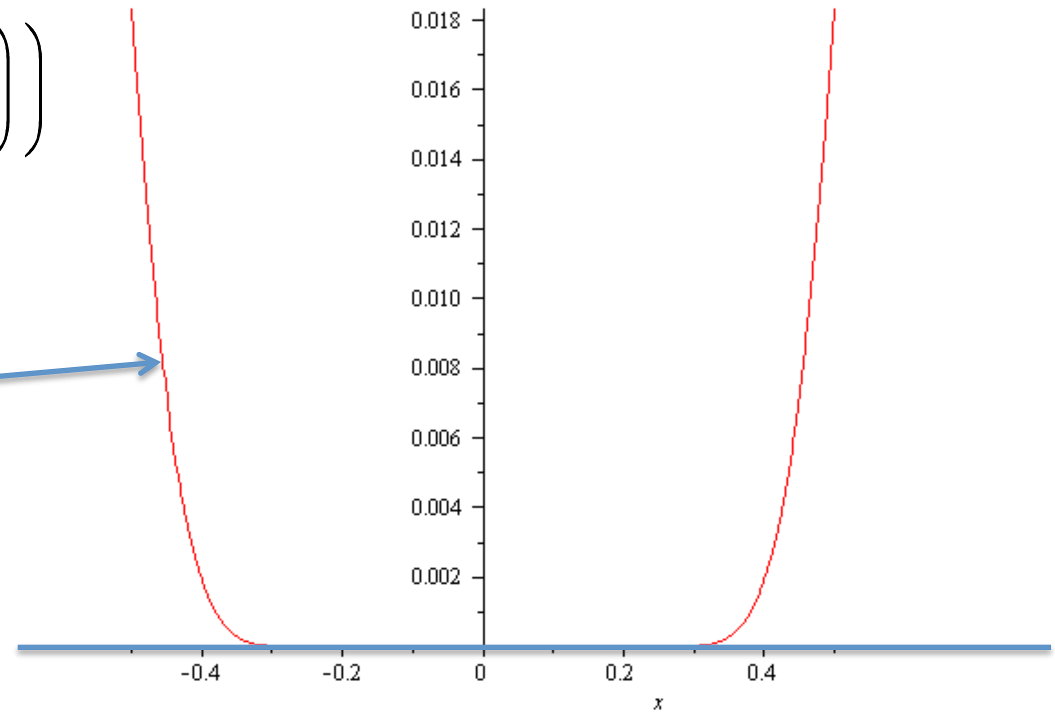
Therefore, the osculating polynomial could be only a straight line that does not converge to the given function.

# How the graphics are?

$$f := piecewise\left(x \neq 0, \exp\left(-\frac{1}{x^2}\right)\right)$$

$$f := \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & otherwise \end{cases}$$

$$plot(f, x = -0.5 \,..\, 0.5);$$




Taylor series

# What else we can find out?

What is desired is that the osculating polynomial had the same position, the same tangent, the same concavity, the same speed of variation of the concavity, and so for all derivatives.

It is clear that in the vicinity of the point $x=0$, the above is not satisfied.

# Conclusions

We must use CAS:

1. For doing many mathematical operations or operations that require much time and effort.

2. For drawing, and displaying properties of complicated functions.

3. For improving the understanding of mathematical objects.

We appreciate your valuable time and we hope to have contributed to the micro-level integration of technology in mathematics education.

**THANK YOU VERY MUCH!**