TIME 2016

Study of a non linear oscillator with CAS through analytical, numerical and qualitative approaches

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Introduction

- According with Blanchard, Devaney and Hall (1999), Differential Equations should be taught holistically with three approaches: analytical, qualitative and numerical.
- There are few programs in Differential Equations with these topics where non linear models appear. (In our case, for the BSc: Applied Mathematics and Computer Science).

"The more realistic is a model in differential equations, the harder it is to find an analytical solution"

Introduction

Q1: Where can professors find any examples more realistic, that involve the three approaches: analytic, qualitative and numerical?

Mickens (2010), in his book "**Truly nonlinear oscillators**" gave good examples of non linear oscillators, but....

Q2: In a first EDO course, is it possible that our students can be prepared to analyze a problem holistically with the three approaches?

CAS could be the bridge in order to achieve it… let´s see a study case!

Many examples of TNL

Examples of second-order, nonlinear differential equations (TNL Oscillators):

$$\ddot{x} + x^3 = 0,$$

$$\ddot{x} + x^{3/5} = 0,$$

$$\ddot{x} + x + x^{1/3} = 0,$$

$$\ddot{x} + x^2 \mathrm{sgn}(x) = 0,$$

$$\ddot{x} + (1 + \dot{x}^2) x^{1/3} = 0,$$

$$\ddot{x} + \frac{1}{x^{1/3}} = 0.$$

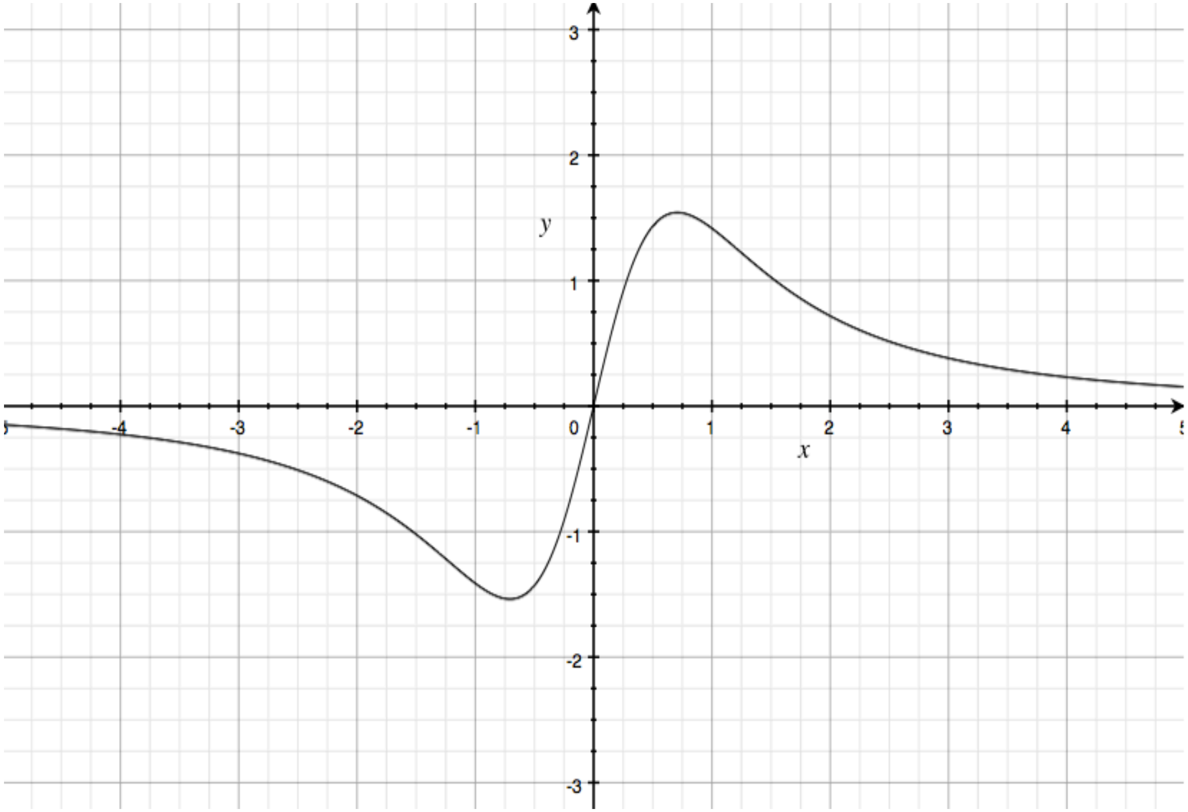
Model

$$\frac{d^2x}{dt^2} + 4x\left(1+x^2\right)^{-\frac{3}{2}} = 0$$

Initial condition: $x(0)=1,\ x'(0)=0$

The nonlinear function

$f(x) = 4x\left(1+x^2\right)^{-\frac{3}{2}}$ is of odd parity, i.e.

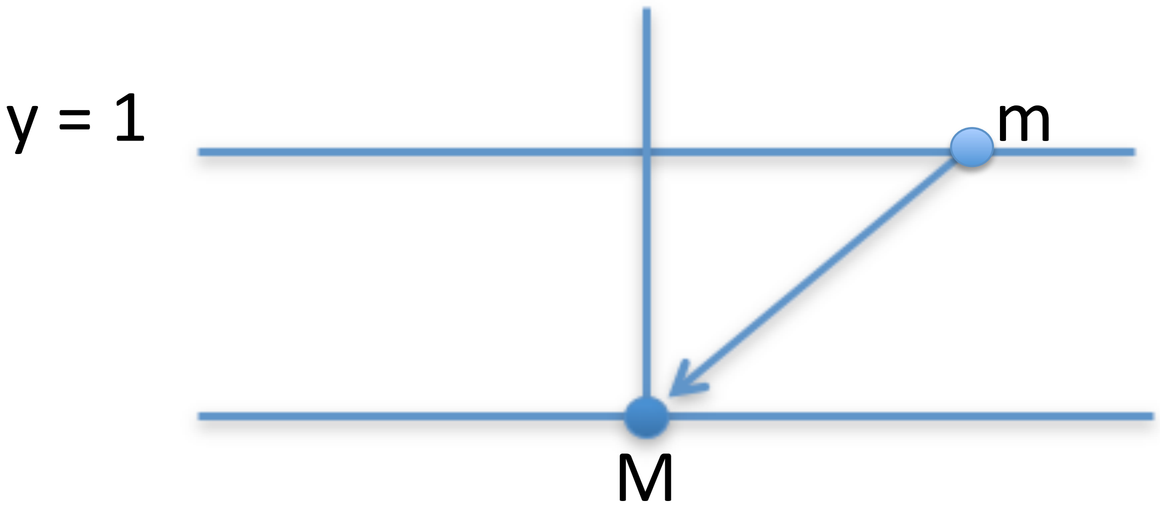


$$f(-x) = -f(x)$$

we assume that f(x) is such that all solutions are periodic

Model in context

A simple interpretation would be consider a particle of mass m which is constrained to move over the line y = 1



and subject to a gravitational interaction with a mass M placed at the origin.

Model in context

The force on the particle in the x direction is given by:

$$F_x = -\frac{GmMx}{\left(1+x^2\right)^{\frac{3}{2}}} \quad F_x = -|F|\cos\theta = -\frac{GmM}{r^2}\cos\theta = -\frac{GmM}{r^2}\frac{x}{r}$$

With $r = \sqrt{1+x^2}$ So, $F_x = -\frac{GmMx}{\left(1+x^2\right)^{\frac{3}{2}}}$

Newton's second law provides us the equation of motion, described as:

$$ma = F_x = -\frac{GmMx}{\left(1+x^2\right)^{\frac{3}{2}}} = m\ddot{x}$$

Let constants: $GM = 4$

Therefore, $\frac{d^2x}{dt^2} + 4x\left(1+x^2\right)^{-\frac{3}{2}} = 0$

Looking for a solution

Three methods exist for carrying out this task:

(1) the use of **analytical methods** in order to find first-integrals,

(2) the use of **qualitative methods** based on examining the geometrical properties of the trajectories in the 2-dim phase-space, and

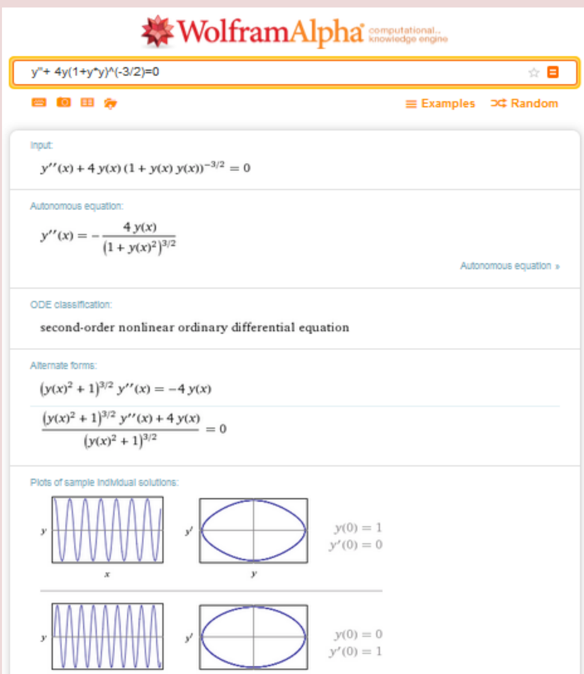
(3) the use **numerical analysis**

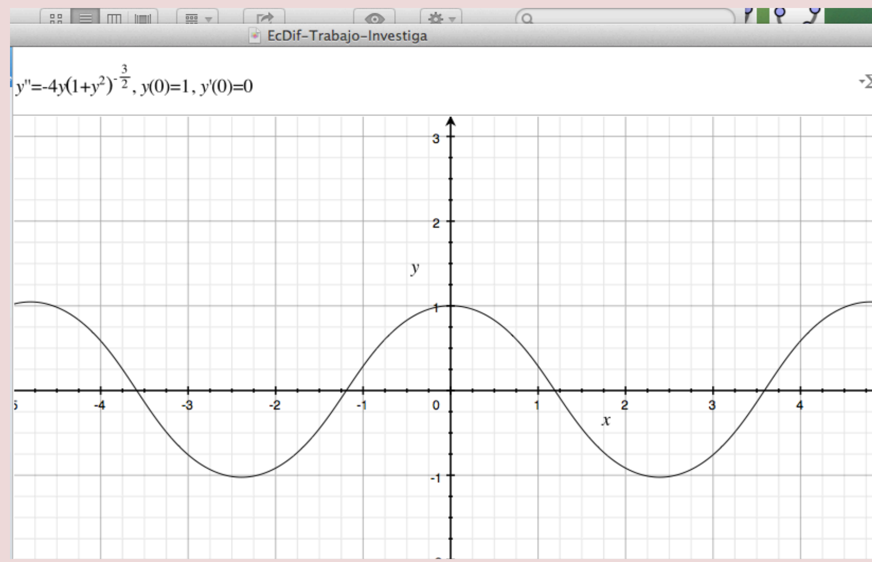
Looking for a solution

"the easy way, without understanding"

It is very common for students tempted to implement immediately the differential equation with some software, using Wolfram alpha or Grapher

WOLFRAM ALPHA (online)	GRAPHER (McIntosh)





Analytical method

First-Integrals

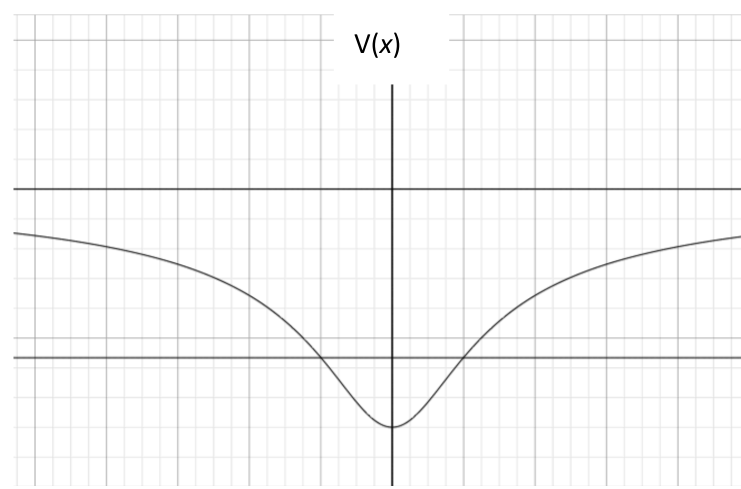
As $f(x) = 4x\left(1+x^2\right)^{-\frac{3}{2}} \neq \begin{cases} f(t) \\ f(\dot{x}) \end{cases}$, it can be shown that energy is conserved.

So, the potential energy is calculated:

$$V(x) = -\int f(x)dx = \int \frac{4x}{\left(1+x^2\right)^{\frac{3}{2}}}dx$$

Let $u = \left(1+x^2\right);\ du = 2xdx$

$$V(x) = \int 2u^{-\frac{3}{2}}du = 2u^{-\frac{1}{2}}(-2) = -\frac{4}{\left(1+x^2\right)^{\frac{1}{2}}}$$



Analytical method

Reduction of order 2 to 1 → Find out a first-integral of the equation

This is done by multiplying $\dot{x} = \frac{dx}{dt}$

Substituting have: $\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\right) = f(x)\frac{dx}{dt} = -\frac{dV}{dx}\frac{dx}{dt} = -\frac{dV}{dt}$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + V\right) = 0$$

Therefore, $\left(\frac{1}{2}\dot{x}^2 - \frac{4}{\left(1+x^2\right)^{\frac{1}{2}}}\right) = E = -2\sqrt{2}$

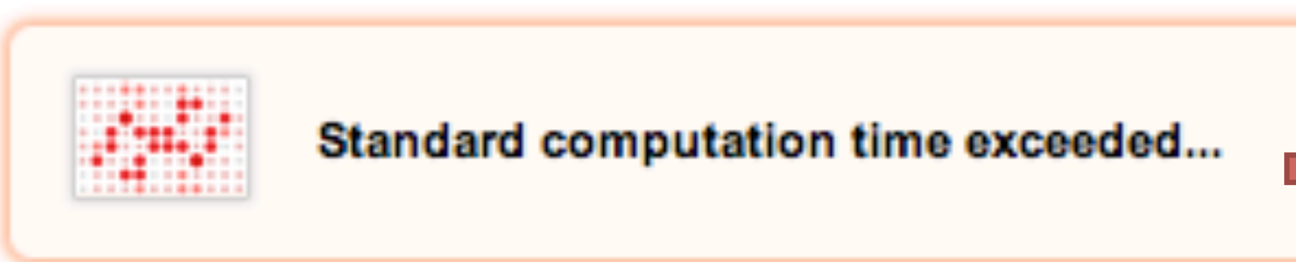
where the initial conditions were used to evaluate E.

Analytical method

So, the 1st-order ODE resulting is: $$\frac{1}{2}\dot{x}^2 = \frac{4}{\left(1+x^2\right)^{\frac{1}{2}}} - 2\sqrt{2}$$

$$\frac{dx}{dt} = \sqrt{\frac{8}{\left(1+x^2\right)^{\frac{1}{2}}} - 4\sqrt{2}}$$

Again, with Wolfram Alpha....



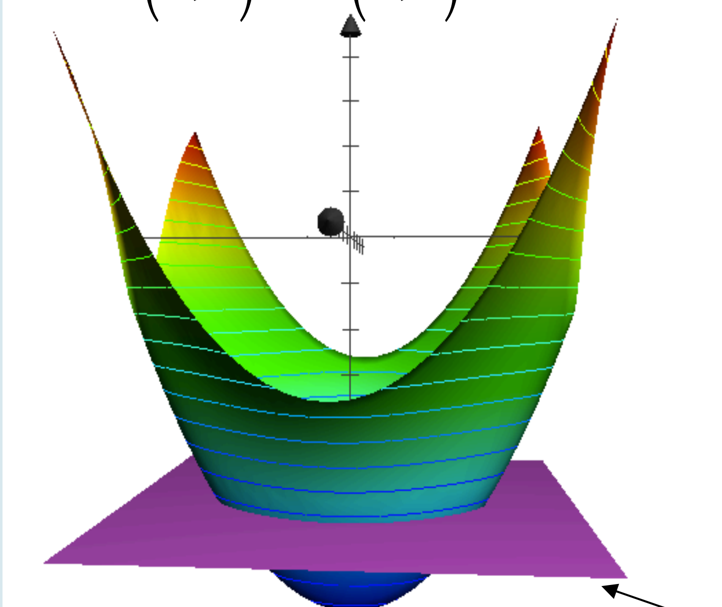
Divergent algorithm!

Connection with qualitative method

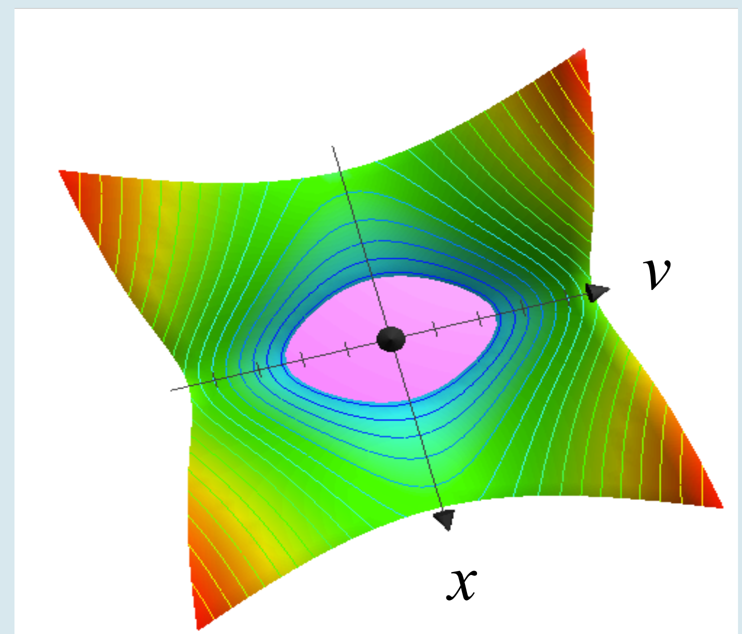
It can be seen that the energy drawn, it represents a closed curve (contour) in the projection in the plane x vs. v=dx/dt

Potential function and the plane representing the constant energy $E=-2\sqrt{2}$	Projection in the plane, with contour lines for different values of energy

$$E = E\left(x,\dot{x}\right) = E\left(x,v\right)$$



$$E=-2\sqrt{2}$$



Phase plane

Qualitative method

The second-order differential equation, may be reformulated to two first-order system equations

$$\dot{v} = f(x), \quad \dot{x} = v$$

i.e.

$$\dot{x} = v$$

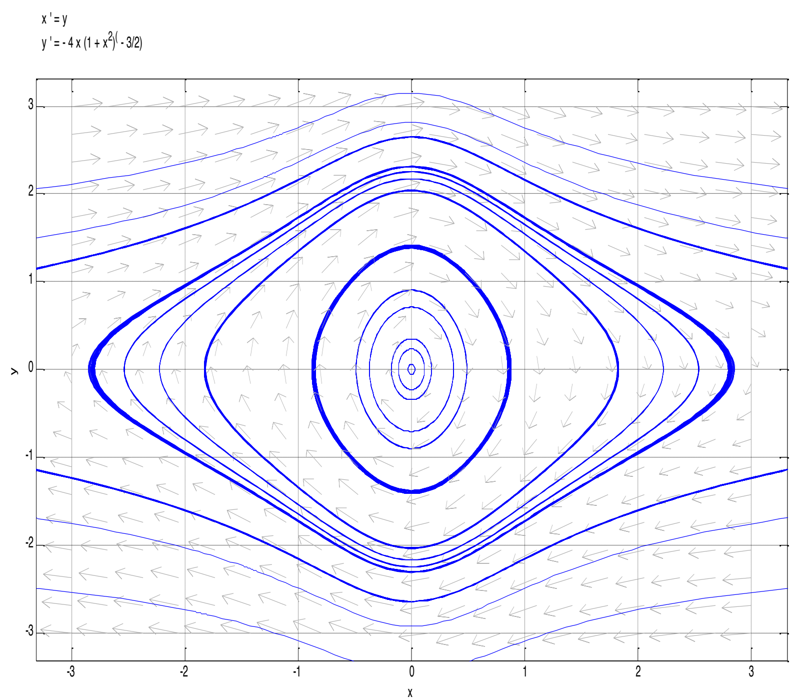
$$\dot{v} = 4x\left(1 + x^2\right)^{-\frac{3}{2}}$$

As usual, the x is interpreted as the position of the particle and the speed v, both dependent functions of time t.

The variables x and v define a 2-dim phase-space which we denote as (x, v) proposed by H. Poincaré.

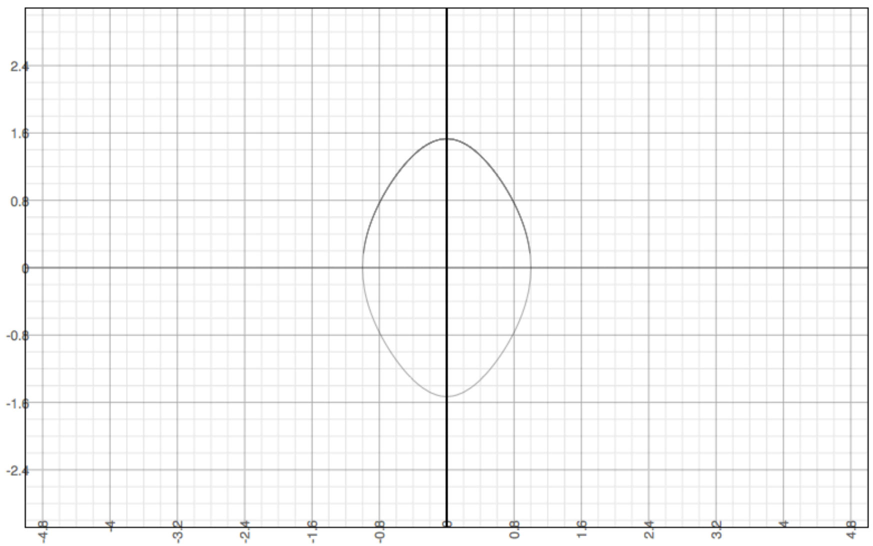
PPLANE - MATLAB

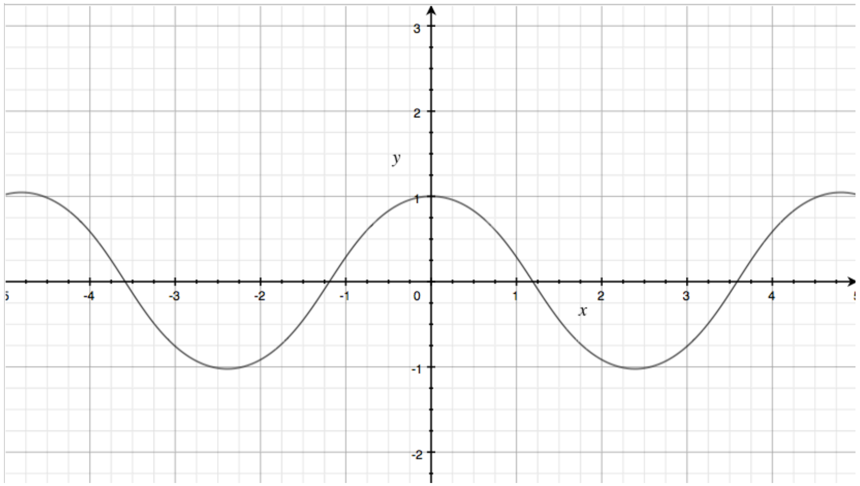
Some of the curves drawn on the **vector field** represent particular solutions when the trajectory associated with the initial values in the phase plane is followed, sometimes called the phase portrait.



Closed Phase-Space Trajectories

In particular, for $(x_0, v_0) = (1,0)$





In this qualitative approach:
it was easy, students can recognized that closed curves in phase-space correspond to periodic solution.

Numerical method

ODE System first order → Application of **Euler algorithm**?

$$x_{n+1} = x_n + v_n h$$

$$v_{n+1} = v_n + F_n h$$

$$t_n = t_0 + nh$$

Disadvantages: It is unstable and not preserve energy.

Vs. **Euler-Cromer algorithm**

$$x_{n+1} = x_n + v_{n+1} h$$

$$v_{n+1} = v_n + F_n h$$

Advantages: It produces stable oscillation, and preserves the total energy produced in each cycle of oscillation.

Euler-Cromer algorithm implemented in Maple

Maple code:

Section 1: Enter data

```
>f:=(t,x)->-4*x(t)*(1+x(t)*x(t))^(-3/2);
```

$$f := (t, x) \rightarrow -\frac{4\,x(t)}{(1 + x(t)\,x(t))^{(3/2)}}$$

Initial conditions

```
>t0:=0.0:
>x0:=1.0:
>v0:=0.0:
>tf:=4.65:
>ics := x(0)=1.0, D(x)(0)=0.0:
```

Section 2: Approach solution

Step 1

```
>h:=(tf-t0)/20.0;
```

$$h := 0.2325000000$$

```
>X[0]:=x0;
V[0]:=v0;
```

$$X_0 := 1.0$$
$$V_0 := 0.$$

```
>f(t0,X[0]);
V[1]:=V[0]+f(t0,X[0])*h;X[1]:=X[0]+V[1]*h;
```

$$-1.414213564$$
$$V_1 := -0.3288046536$$
$$X_1 := 0.9235529180$$

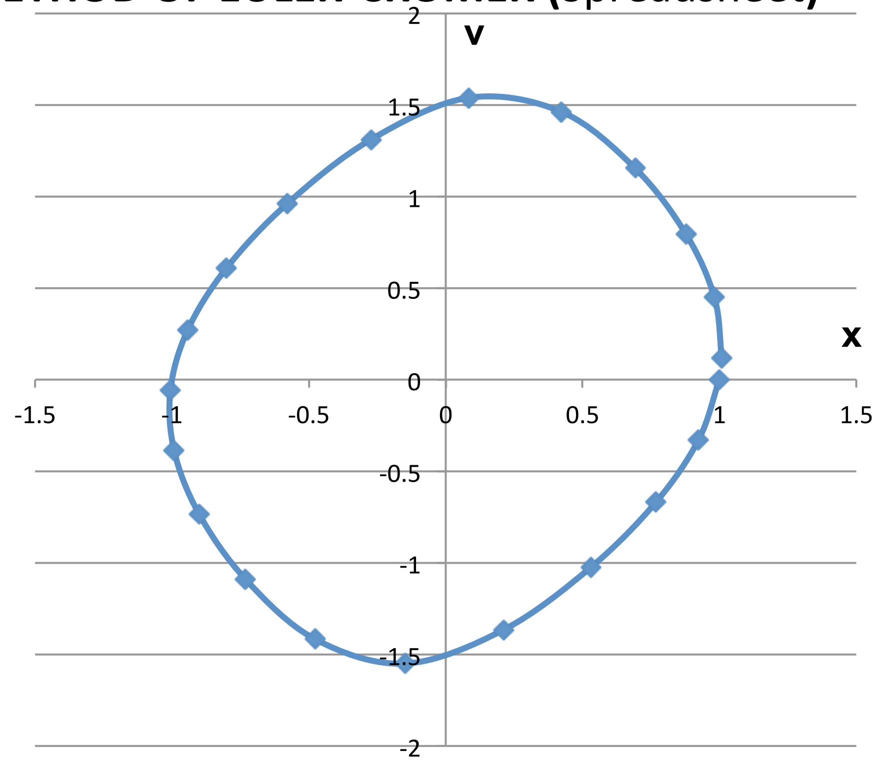
Total iterations: 20

Results: Euler-Cromer algorithm

The result for n = 20 iterations shown in the following table with the calculation of energy at each step

n	Xn	Vn	En	Δ(E)
0	1	0	-2.82842712	
1	0.92355292	-0.32880465	-2.88446031	-0.05603318
2	0.76793374	-0.6693298	-2.94848228	-0.06402197
3	0.52947309	-1.02563719	-3.00909619	-0.06061391
4	0.2119881	-1.36552687	-2.98071039	0.0283858
5	-0.14840912	-1.55009556	-2.7552659	0.22544449
17	0.69177505	1.15383861	-2.62391681	-0.00840372
18	0.87684401	0.79599553	-2.69075226	-0.06683545
19	0.98132137	0.44936502	-2.75400039	-0.06324813
20	1.00864811	0.11753434	-2.80931624	-0.05531585

NUMERICAL PHASE CURVE ASSOCIATED WITH THE METHOD OF EULER-CROMER (Spreadsheet)



Students can check is the same closed curve!

Conclusions / Reflections

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- Our purpose is linking research-teaching with cutting edge topics, as non linear differential equations.

- The qualitative technique is more powerful since it may be applied in all situations, in comparison with the analytical technique.

- The goal is to show that either all or some of the trajectories in the phase-plane are closed. Since closed trajectories correspond to periodic solutions, the existence of periodic solutions was established.

Conclusions / Reflections

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- Visualization helps intuitive understanding.

- The learning-teaching process is shorter and dynamic in order to visualize that the vector field, potential function, phase plane, etc., associated with the differential equations.

- A good combination of CAS could give us the three analysis. Such as: Maple, MatLab, Graph, and Excel.

- We say that how much software that we have to use depends on the dynamics of the class.

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Thank you!

¡Gracias!