Differential equations as a teaching topic in school?

1. Introduction

Physics teachers have treated already ordinary differential equations in their lessons before computer algebra was available. They did so for good reasons.

1. Finding and solving diffential equations is one of the most powerful tools of physisists in order to understand physical processes, to discover their laws and to make predictions.
2. Many physical laws are formulated as differential equations, e.g. the Newton’s second law, the law of induction, the wave equation or the Schrödinger equation.

High school students in high schools are neither supposed to know the theory of differential equations nor are they taught to solve the equations. They need help from a teacher who suggest very often a function as a purported solution and asks to verify the claim. When a CAS is used the computer can be asked for the solution instead of the teacher. But a CAS can do more. It can solve differential equations also numerically. It opens the door to a variety of other processes in nature. Physics in school was and is essentially restricted to linear phenomena because in general only linear problems could be solved mathematically. If a CAS is available nobody has to distinguish between linear, nonlinear or even chaotic processes. The numeric solver treats all kinds of ODE in the same way. It replaces the continuous problem by a discrete substitude and operates on the latter by iterative processes starting from the initial condition.

In order to introduce ordinary differential equations in school the students must learn:

- How to derive the differential equations of physical processes.
- How solutions of differential equation may be approximated numerically
- How differential equation of order 2 and larger may be converted into a system of first order differential equations
- How to apply the tools of TI-89/92+ to obtain and process the solution of ordinary differential equations.

I would like to show how this could be realized in physics and/or math lessons. All my examples deal with differential equations of oscillations. I will begin with harmonic oscillations. Then I will pass to the physical pendulum as an example of a nonlinear oscillation. Finally chaotic oscillations are simulated.

Exact solution of differential equations with *DeSolve()*

When investigating oscillations in physics lessons it makes sense to start with harmonic oscillations. The differential equation of free undamped oscillations of e.g. a spring-mass system is given by:

$$\ddot{y} = -\frac{d}{m} \cdot y, \quad (1)$$

where d is the spring constant, m the mass.

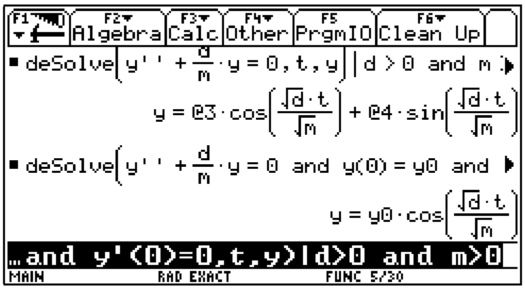


Fig. 1

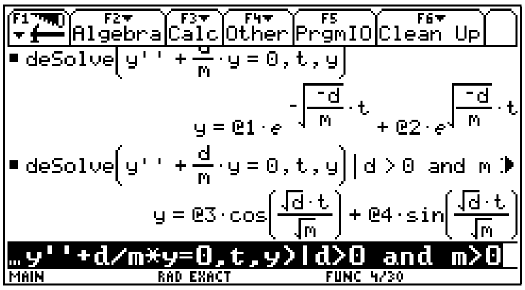


Fig. 2

When the DeSolve() command is applied to equation (1) the general solution is written in Terms of exponential functions. (second row in fig.1). Students are not used to exponential functions with imaginary arguments at this time and therefore only solutions with trigonometric functions for the periodic motion are applicable. This can be produced by means of the „*with*“ operator as it can be seen in row 3 of fig.1. A particular solution is achieved when the initial or boundary conditions are added to the „*DeSolve()*“ command (row 3 and 4 in fig.2). One can proof easily that the function y in fig.2 is a solution of (1) by the operations shown in fig.3.

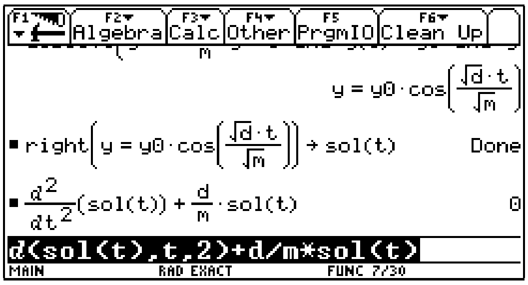


Fig. 3

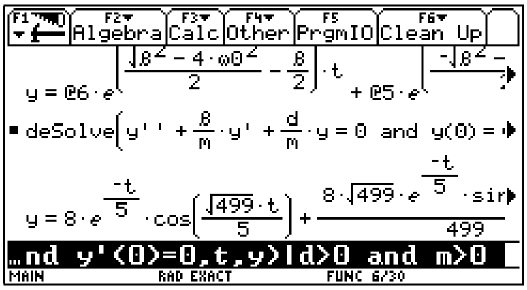


Fig. 4

From the solution

$$y = y_0 \cdot \cos(\sqrt{\frac{d}{m}} \cdot t) \quad (2)$$

the basic properties of the harmonic oscillations can be deduced in class.

Adding a friction force proportional to the velocity to equation (1),

$$\ddot{y} = -\frac{d}{m}\cdot y - \frac{\beta}{m}\cdot \dot{y}\,, \quad (3)$$

then again a solution with exponential functions appear (fig.4, first row) which in this case cannot be written in terms of trigonometric functions. In school therefore only solutions with defined parameters y_0, d and m may be studied. The result is shown in the last row of fig.4.

Numerical solutions of differential equations

The numerical solution of differential equations is gotten by solving the corresponding difference equations by iterations starting from an initial or boundary value. Only first order differential equations can be converted into difference equations. For that reason differential equations of order 2 or higher must be transformed into a system of first order equations. This formal transformation of a second order differential equations $\ddot{y} = f(x, y, \dot{y}, g(t))$ can be done by replacing y by y_1 and $\dot{y}$ by y_2 which yields:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= f(t, y_1, y_2, g(t)) \end{aligned}.$$

This formal process does not spark students imagination. For them it is easier to express e.g. the equation (3) as an equation for velocity v, a well known quantity, by using $\dot{y} = v$ and $\ddot{y} = \dot{v}$. Then the system of first order equations defines the velocity and the derivative of velocity.

$$\begin{aligned} \dot{y} &= v \\ \dot{v} &= -\frac{d}{m}\cdot y - \frac{\beta}{m}\cdot v \end{aligned}$$

The geometric picture of the phase space might be helpful conceptually, because it reduces the solution of an ODE to the passage from the description of a flow given by a vector field to a flow in terms of appropriately parametrized streamlines on the same field.

TI-92 needs the variable y_1 and y_2 instead of y and v and one has to type at the Y=Editor:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -\frac{d}{m}\cdot y_1 - \frac{\beta}{m}\cdot y_2 \end{aligned}. \qquad (4)$$

One has the choice to solve the system aproximately by using the Euler or the Runge-Kutta method. Specially the Euler method is very critical with respect to the step width of the discretization. If it is too large an exponentially increasing function is obtained as a solution of (3) what is definitely wrong. In order to make sure that a reasonable approximation is reached with the Runge-Kutta method the numerical and the exact solution of equation 3 is displayed in fig.5.

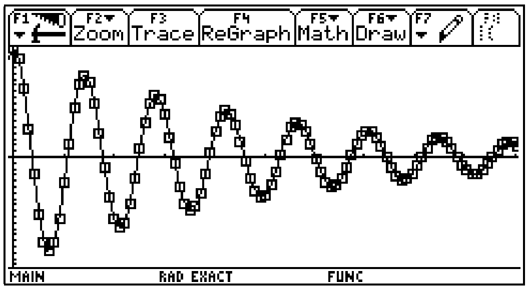


Fig. 5

There are no obvious differences between the approximation and the exact solution.
Beside the harmonic oscillations it is important to discuss also oscillations which are nonharmonic.

The physical pendulum with large angular displacements is an often cited example because the experiment is easy to realize. In the demonstrated experiment the angle of a swinging bar is measured by means of a potentiometer at the axis of rotation as it can be seen in the sketch in fig.6. The voltage at the potentiometer is recorded by means of the computer based analog digital converter CBL.

The data are displayed with repect to time in fig.7. The angle at the cursor position is 160°. It is obvious that the period of the oscillation decreases with decreasing angular displacement.

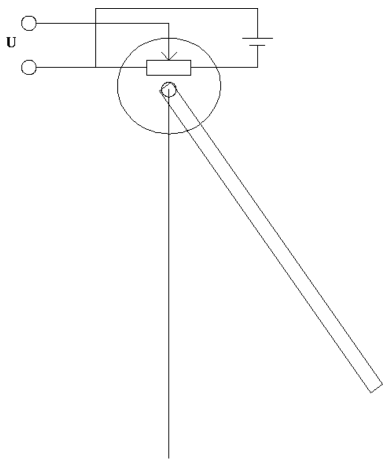


Fig. 6

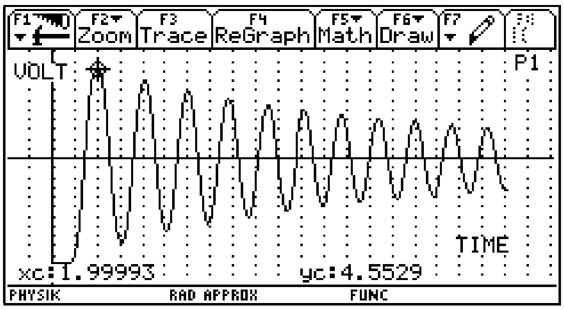
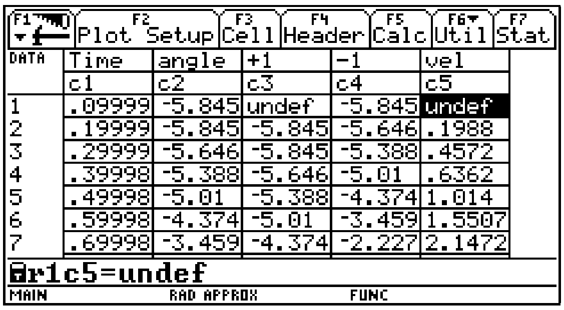


Fig. 7



DATA	Time	angle	+1	-1	vel
	c1	c2	c3	c4	c5
1	.09999	-5.845	undef	-5.845	undef
2	.19999	-5.845	-5.845	-5.646	.1988
3	.29999	-5.646	-5.845	-5.388	.4572
4	.39998	-5.388	-5.646	-5.01	.6362
5	.49998	-5.01	-5.388	-4.374	1.014
6	.59998	-4.374	-5.01	-3.459	1.5507
7	.69998	-3.459	-4.374	-2.227	2.1472

r1c5=undef

Fig. 8

By using the *„Trace“* option of the TI-92+ the period can be calculated from the positions of the maxima and minima. In fig.8 the data of the CBL are loaded in a table, the first columns show time and angle. In column 5 the angular velocity is calculated. Applying the *„Shift()“*

command to the data in column 2 the shifted angles φ_{i+1} (column 3) and φ_{i-1} (column 4) are listed. From these columns the angular velocity

$$\dot{\varphi} = \frac{\varphi i_{i+1} - \varphi_{i-1}}{\Delta t}$$

can be determined numerically (column 5). This allows to display the phase portrait of the oscillation (fig 9). On the right hand side of the graph the angles can only be 160° at maximum because the potentiometer only covers a range of 160°.

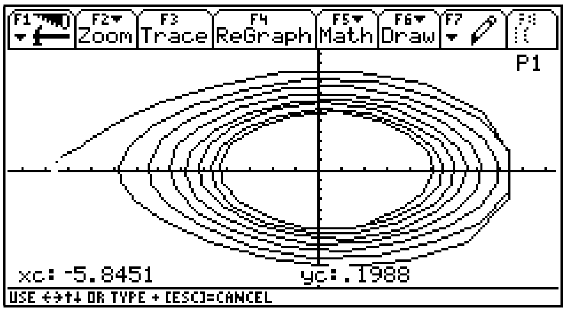


Fig. 9

The nature of the pendulum can be investigated more thoroughly when the solutions of the differential equation are considerered for different cases. The differential equation of the physical pendulum with a friction force proportional to the angular velocity is given by:

$$\ddot{\varphi} = -\frac{m \cdot g \cdot s}{J} \cdot \sin\varphi - \frac{\beta}{J} \cdot \dot{\varphi}\,. \ (5)$$

where first the term describes the torque of the bar and the second the friction force. In order to study the relation between the period of oscillation and the anglular displacement, equation 5 has been solved for $\beta = 0$. The results are shown in fig.10 to 12.

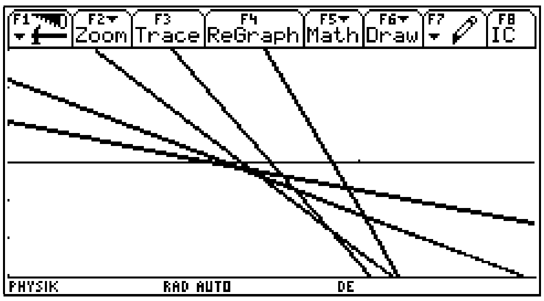


Fig. 10

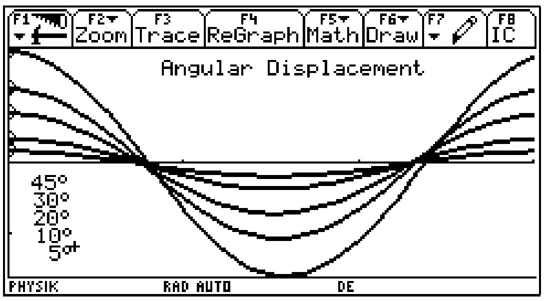


Fig. 11

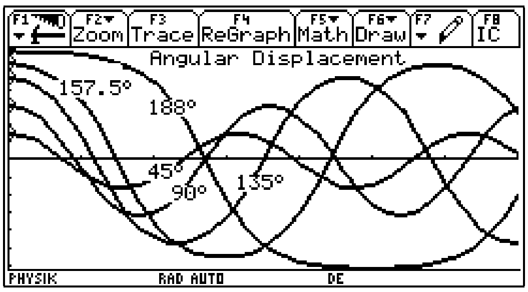


Fig. 12

For initial angles exceeding 90° it is obvious that the oscillation is not sinusoidal. Also the oscillation period clearly depends on the amplitude. For angles below 45° the rate of change is smaller as it is shown in fig.11 and fig.12 where the region near the zero on the t-axis is enlarged displayed. The relation between frequency and displacement can be investigated qualitatively from these graphs. Finally the observed oscillation of fig.7 is modeled by the differential equation 5. The damped oscillation is displayed in fig.13.

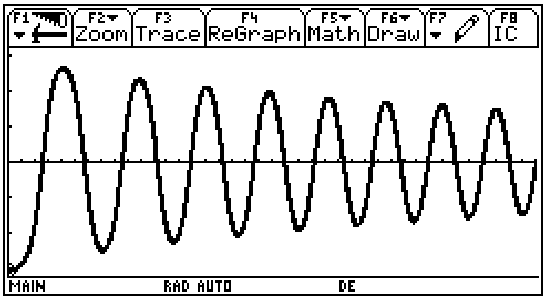


Fig. 13

The last oscillator which is to be investigated here is a rotational pendulum with an unbalance. The swinging wheel in fig.14 has two stable states. When it is exited by a periodical force it has 3 degrees of fredom and it may therefore exhibit chaotic oscillations. Therefore it is one of those elementary experiments to illustrate deterministic chaos.

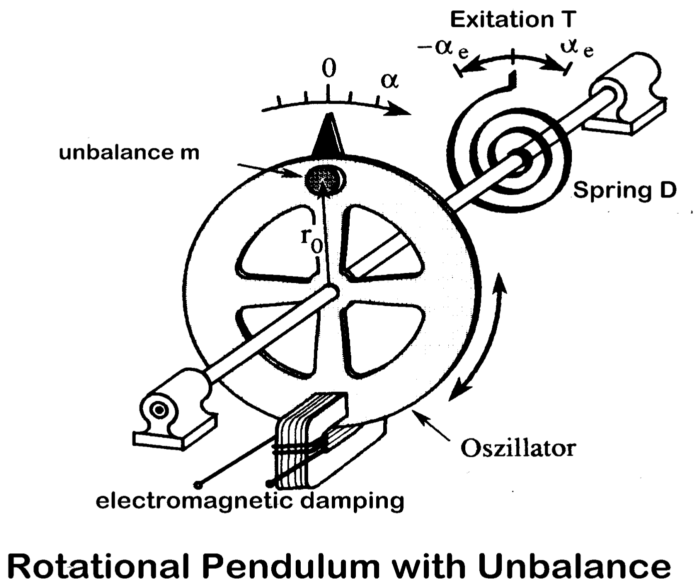


Fig. 14

The differential equation is also well known and can be solve by the ode solver of TI-89/92+. The according differential equation is :

$$\ddot{\varphi} = -\frac{D}{J}(\varphi - \varphi_e \cdot \sin \omega_e t) + \frac{m \cdot g \cdot r_0}{J} \cdot \sin \varphi - k_1 sign \dot{\varphi} - k_2 \cdot I^2 \cdot \dot{\varphi} . \qquad (6)$$

Here the first term of right side of the equation describes the driving force of the oscillation. The second term is the torque of the unbalance. The last two term give the constant friction force and a friction force proportional to the angular velocity which depends on the variable current through the damping magnet.

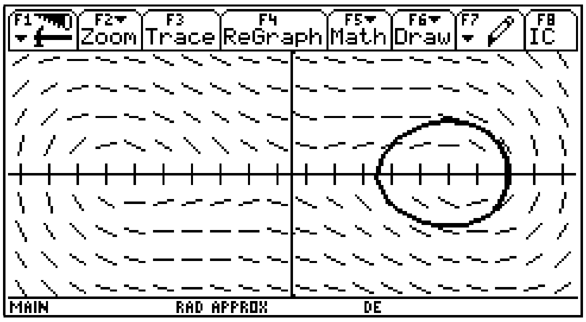


Fig. 15

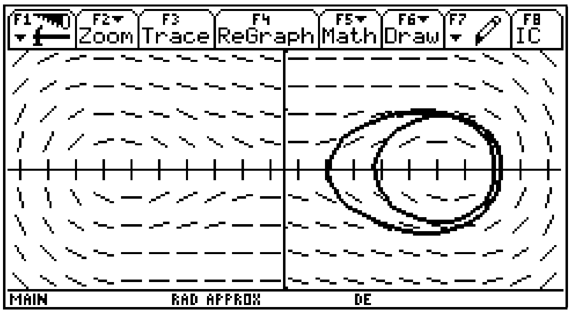


Fig. 16

Fig. 15 and 16 demonstrates the bifurcation scenario for different currents I. In fig.17 chaotic motion is displayed versus time and fig.18 shows then corresponding phase portrait.

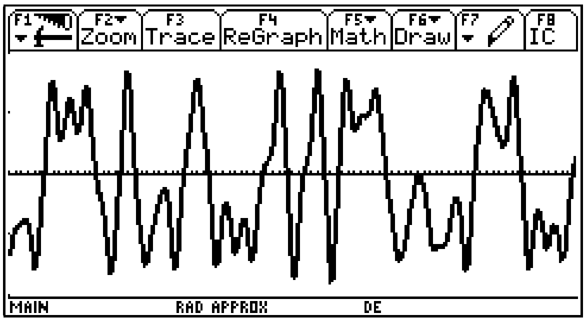


Fig. 17

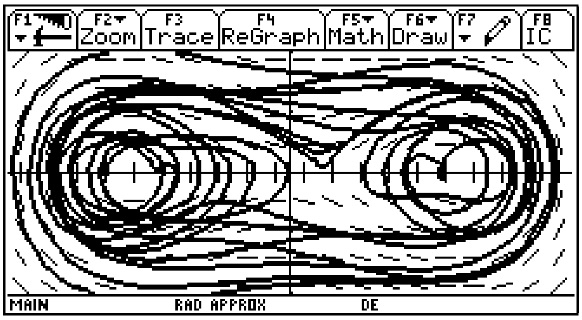


Fig. 18

4. Conclusion

I tried to give you an idea what can be done with the differential equation solvers, DeSolve() and the numeric solver of the TI calculators.
The introduction of a CAS in schools gives the chance to add new topics to the classical curriculum in mathematics and physics because students spend less time with numerical work. It is possible now to model real world processes by solving differential equations in math or physics lessons. Is it also worthwhile to do?