Are CIRCULAR functions TRIGONOMETRIC or REAL?

Giora Mann and Nurit Zehavi
Levinsky College of Education, Israel   Weizmann Institute of Science, Israel

Introduction: When you ask a student before learning calculus about sin(0.7) – he may answer with a question: "You mean 0.7 degrees?". Nothing like that will happen if you ask about values of other REAL functions! Why do most students NOT see the circular functions (sin, cos, tan, etc) as an integral part of the family of REAL functions? The answer is simple (if primitive…) "If sin(x) is a TRIGONOMETRIC function, how could it be REAL at the same time?" If you want your students to accept sin(x) as another REAL function you should introduce it as one, otherwise, the first impression - that without angles and triangles you don't speak about sin(x) - will prevail.

History is the source of the above problem, but gives us also the clue to the solution. Astronomical needs gave rise to the first TRIGONOMETRIC TABLES. Claudius Ptolemy (c. 100-178 C.E.) introduced in his *Almagest* (which contains a complete mathematical description of the Greek model of the universe, and was a culmination of Greek astronomy [1]) a table of chords for all arcs from ½ degree to 180 degrees in intervals of ½ degree. Since arcs in heaven are not assessable angles are used and hence degrees. On the other hand, Archimedes' (287-212 B.C.E.) method for computing the circumference of the circle [2] can be used to compute the chord for any arc in a unit circle, or compute the arc corresponding to any given chord of the unit circle. CAS enables us to carry out the needed computations in a way that a common student can follow and you don't need an Archimedes to do it. When you introduce sin(x) as a CIRCULAR function it is from the start a REAL function.

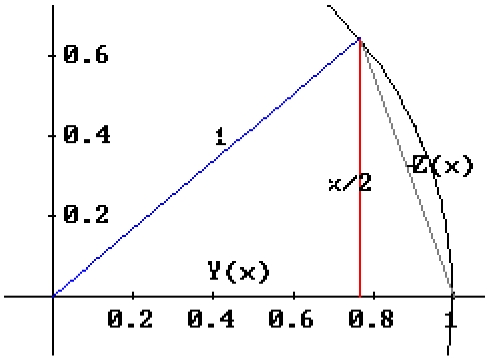
The following unit was developed in the Weizmann Institute for Science for math teachers as a first step towards the needed changes in the curriculum which will integrate the circular functions into the family of real functions.

[1] A HISTORY OF MATHEMATICS, Victor J. Katz pp. 136-137.
[2] A HISTORY OF MATHEMATICS, Victor J. Katz pp. 101-103.

Following is the introduction of the inverse of sin(x), with some didactical remarks. Next will come the introduction of sin(x), and the rest follows.

The function ASIN(x)

Archimedes used the Pythagorean theorem twice in the following triangle to iterate from a given chord – x - (in his case the diameter) to the corresponding arc:



$$Y(x) := \sqrt{\left(1 - \frac{x^2}{4}\right)}$$

$$Z(x) := \sqrt{\left(\frac{x^2}{4} + (1 - Y(x))^2\right)}$$

$$Z(x) := \sqrt{(2 - \sqrt{(4 - x^2)})}$$

At this point in time it seemed a good idea to simplify Z(x). We are going to regret it shortly. Next we define the nth approximation to the arc corresponding to a given chord x in the unit circle:

$$ARC(x, n) := 2^n \cdot ITERATE(Z(t), t, x, n)$$

A few approximations obtained in **exact** mode are:

$$ARC(2, 4) = 3.13656$$

$$ARC(2, 5) = 3.14037$$

$$ARC(2, 6) = 3.14132$$

$$ARC(2, 7) = 3.14158$$

$$ARC(2, 8) = 3.14245$$

$$ARC(2, 9) = 3.14245$$

It takes 60 seconds to compute ARC(2,10) and get an unexpected result:

$$ARC(2, 10) = 3.16227$$

We change to **approximate** mode and get even a worse result (much faster!):

$$ARC(2, 9) = 3.14122$$

$$ARC(2, 10) = 0$$

It seems that mistakes are accumulating much faster! – Probably because we simplified Z(x). So, let's cancel this last simplification by:

$$Z(x) :=$$

$$Z(x) := \sqrt{\left(\frac{x^2}{4} + (1 - Y(x))^2\right)}$$

And find that even after 20 iterations we get a nice result for π:

$$ARC(2, 10) = 3.14158$$

$$ARC(2, 20) = 3.14158$$

The ARC function is closely related to the ASIN function of DERIVE. Let's define the circular function ARCSIN(x,n) as follows:

$$\text{ARCSIN}(x, n) := \frac{\text{SIGN}(x)}{2} \cdot \text{ARC}(2 \cdot |x|, n)$$

A few comparisons will do now:

$$\text{ARCSIN}(0.8, 20) = 0.927293$$
$$\text{ASIN}(0.8) = 0.927295$$
$$\text{ARCSIN}(-1, 20) = -1.57079$$
$$\text{ASIN}(-1) = -1.57079$$

The function SIN(x)

In order to compute the chord corresponding to a given arc we use the relation between z and x backwards:

$$z = \sqrt{(2 - \sqrt{(4 - x^2)})}$$
$$z^2 = 2 - \sqrt{(4 - x^2)}$$
$$z^2 - 2 = - \sqrt{(4 - x^2)}$$
$$(z^2 - 2 = - \sqrt{(4 - x^2)})^2$$
$$z^4 - 4 \cdot z^2 + 4 = 4 - x^2$$
$$x^2 = 4 \cdot z^2 - z^4$$

Since we assume that both x and z are positive we get:

$$x = \sqrt{(4 \cdot z^2 - z^4)}$$

We assume now that the original arc (x) was divided to 2^n equal arcs. We may further assume that each one of the small arcs is very close to its corresponding chord, so that $\frac{x}{2^n}$ is the nth approximation in the original sequence. To get the zero approximation (the desired chord), all we have to do is to iterate backwards n times using the last formula.

$$\text{CHORD}(x, n) := \text{ITERATE}\left(\sqrt{(4 \cdot z^2 - z^4)}, z, \frac{x}{2^n}, n, 0, -1\right)$$

Here are some examples demonstrating this formula:

$$\mathrm{CHORD}(\pi, 4) = 2$$

$$\mathrm{PrecisionDigits} := 12$$

$$\mathrm{CHORD}(\pi, 4) = 1.99999$$

$$\mathrm{CHORD}(\pi, 6) = 2$$

$$\mathrm{PrecisionDigits} := 18$$

$$\mathrm{CHORD}(\pi, 6) = 1.99999$$

$$\mathrm{CHORD}(\pi, 8) = 1.99999$$

$$\mathrm{CHORD}(\pi, 10) = 2$$

Obviously, more precision digits imply more iteration to get a better result.

$$\mathrm{CHORD}\left(\frac{\pi}{2}, 8\right) = 1.41421$$

$$\sqrt{2} = 1.41421$$

Here, we get again the expected result. We are now ready for a definition of the **SINE** function:

$$\mathrm{SINE}(x, n) := \frac{1}{2}\cdot \mathrm{CHORD}(2\cdot x, n)$$

Let’s see a few examples:

$$\mathrm{SINE}(1, 12) = 0.841472$$

$$\mathrm{SIN}(1) = 0.841470$$

$$\mathrm{SINE}(2, 12) = 0.909295$$

$$\mathrm{SIN}(2) = 0.909297$$

$$\mathrm{SINE}(3, 12) = 0.141117$$

$$\mathrm{SIN}(3) = 0.141120$$

$$\mathrm{SINE}(4, 12) = 0.756808$$

$$\mathrm{SIN}(4) = -0.756802$$

$$\mathrm{SINE}(-1, 12) = 0.841472$$

$$\mathrm{SIN}(-1) = -0.841470$$

As one can see this definition applies only if $0 < x < \pi$. We need take care of the period 2π, and be sure that the CHORD function is applied only if $0 < x < \pi$:

$$\mathrm{SINE}(x, n) := -\frac{\mathrm{SIGN}(\mathrm{MOD}(x, 2\cdot\pi) - \pi)\cdot \mathrm{CHORD}(2\cdot \mathrm{MOD}(x, 2\cdot\pi), n)}{2}$$

Here are some results:

$$\text{SINE}\left(\frac{\pi}{4}, 12\right) = 0.707106$$

$$\text{SIN}\left(\frac{\pi}{4}\right) = 0.707106$$

$$\text{SINE}\left(- \frac{13\cdot\pi}{6}, 12\right) = -0.5$$

$$\text{SIN}\left(- \frac{13\cdot\pi}{6}\right) = -0.5$$

$$\text{SINE}(20\cdot\pi, 12) = 0$$

$$\text{SIN}(20\cdot\pi) = 0$$

$$\text{SINE}\left(\frac{17\cdot\pi}{2}, 12\right) = 1$$

$$\text{SIN}\left(\frac{17\cdot\pi}{2}\right) = 1$$

One can see that we have built quiet a good approximation to the sin function, based on the idea of the chord function.

CONCLUSION

We saw in this paper how the functions SIN and ASIN can be computed as REAL functions using some basic geometric and algebraic notions. Since the geometry of the circle is the one used the name CIRCULAR functions fits the two functions and functions derived from them (COS, TAN, ACOS, ATAN):

$$\text{SIN}\left(\frac{\pi}{2} - x\right) = \text{COS}(x)$$

$$\frac{\text{SIN}(x)}{\text{SIN}\left(\frac{\pi}{2} - x\right)} = \text{TAN}(x)$$

$$\text{ACOS}(x) = \frac{\pi}{2} - \text{ASIN}(x)$$

$$\text{ASIN}\left(\frac{x}{\sqrt{(1 + x^2)}}\right) = \text{ATAN}(x)$$

If we take this venue (taking advantage of the DERIVE environment) we may delete the confusion of students looking for angles and triangles whenever facing a SIN.