Kepler's Ellipses: Challenges from Compasses to Computer Algebra

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DERIVE solves non-solved problem in physics teaching.
In each physics textbook for secondary school and university you will find formulas and plots for time dependence of displacement, velocity and acceleration for most basic motions: uniform motion, uniform accelerated motion, harmonic motion, but not for planetary motion. Newton has no mathematical tools like ***DERIVE***, so he has advised numerical approximation, which is good enough for research, but not suitable for teaching. It is time to solve this problem in teaching physics. With ***DERIVE*** we can do this.

A well known legend tells us that Newton discovered the law of Universal Gravitation by observing a falling apple. The fact is that in PRINCIPIA [1] (1687) he documented the law of attractive force between two masses. Two masses -- for example two planets, or the Earth and an apple -- will attract each other with a force which is inversely proportional to the square of the distance between them. The two body problem can be reduced to a one body problem. Newton set the main goal of mechanics: to find time dependence of coordinates for moving body: $x(t)$, $y(t)$, $z(t)$. However, in 300 years the special case of motion in a circular orbit:

$$x(t)=R\cdot\cos(\omega t),\ y(t)=R\cdot\sin(\omega t),\ z(t)=0,$$

where R is the distance between the point mass and center of force is the only exact solution to have been found.
To find $x(t)$, $y(t)$ for the more general case of an elliptical it is necessary to use approximate numerical methods or a numerical solution of the differential equation, for example, with the Runge-Kutta method.
In standard astronomical books the planetary motion is described in polar coordinates by the equations [2]:

$$r=\frac{p}{1+\varepsilon\cdot\cos(\varphi)},$$

$$tg\frac{\varphi}{2}=\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\cdot tg\frac{E}{2},$$

$$E-\varepsilon\cdot\sin(E)=\frac{2\pi t}{T}$$

This solution is too difficult for analyzing in secondary school or university standard physics course. I will show, that computer algebra ***DERIVE*** solves this problem in physics teaching completely. With ***DERIVE*** you can process the problem analytically, visualize graphically, solve numerically.

Another method which can be employed in looking for the exact solution of the equation of motion in inverse square force field is a parametric set of expressions: $t(E)$, $x(E)$, $y(E)$. Newton has found such solution – time cycloid:

$$t=\frac{2\pi}{T}(E-\varepsilon\cdot\sin(E))$$

$$r=a(1-\varepsilon\cdot\cos(E))$$

Newton used geometrical presentation of cycloid ([2], Fig. 62). May be this was a reason, why he has written [2]: “But since the description of this curve is difficult, a solution by approximation will be preferable” (… for 300 years!).

In Max Born famous book [3] we will find the most correct set of parametric solution for planetary motion:

$$
\begin{aligned}
&r = a\cdot(1-\varepsilon\cdot\cos u),\\
&\xi = a\cdot(\cos u-\varepsilon)\\
&\eta = a\sqrt{1-\varepsilon^2}\,\sin u\\
&2\pi\nu\cdot t = u-\varepsilon\cdot\sin u
\end{aligned}
$$

Max Born wanted the solution in traditional form $r(t)$, $\xi(t)$, $\eta(t)$. To do this, he used the Laplace method, developing the expressions in Fourier series.

Let us show, that in the world of computer algebra parametric solution is suitable for teaching purposes.

We will study Kepler's ellipses with initial coordinates x=R, y=0 and velocity $|\mathbf{v}_{-}+\mathbf{v}_{+}| = v_{+} - v_{-}$ changing from $v_0 = 2\pi R/T$ (circular orb, $\varepsilon = 0$) to 0 (free fall on force center, $\varepsilon = 1$).

$$
\begin{aligned}
&t = \frac{T}{2\pi}\frac{(E+\varepsilon\cdot\sin(E))}{(1+\varepsilon)^{\frac{3}{2}}},\\
&x = \frac{R}{1+\varepsilon}(\varepsilon+\cos(E)),\\
&y = R\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\sin(E)
\end{aligned}
$$

where R is the radius of circular orbit and T period of motion on circular orbit.

With one ***DERIVE*** expression

VECTOR([√((1-ε)/(1+ε))SIN(E),(ε+COS(E))/(1+ε)],ε,[0,0.5,0.8,0.9,0.99])

we can plot (after Simplification) orbits in scaled coordinate system [*x/R, y/R, t/ T*] (Fig. 1).

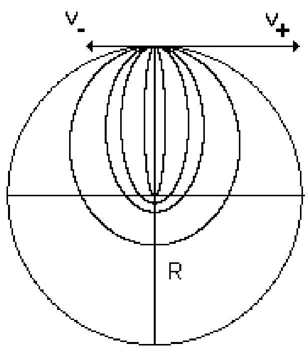

Fig. 1. Kepler's ellipses for ε = 0, 0.5, 0.8, 0.9, 0.99.

From point of view of CAS the given parametric set of expressions is exact solution of planetary motion. We can apply algebra, calculus and plot commands. It is not necessary to transform the parametric expressions in $x(t)$, $y(t)$, $r(t)$ to analyze the planetary motion.

With ***DERIVE*** commands

[vx:=DIF(x,E)/DIF(t,E), vy:= DIF(y,E)/DIF(t,E), ax:= DIF(vx,E)/DIF(t,E), ay:= DIF(x,E)/DIF(t,E)]

we can find velocity and acceleration as parametric expressions and after then show, that Kepler's laws results from parametric solution of planetary motion:

1. Simplification of $\sqrt{(x^2 + y^2)} + \sqrt{(2a\varepsilon + x)^2 + y^2}$ gives an invariant $2R/(1+\varepsilon)=2a$, where $a=R/(1+\varepsilon)$.
Kepler's first law: The planet describe ellipses with the Sun at one focus.

2. Simplification of $(1/2)|\mathbf{r}\times\mathbf{v}| = (1/2)[x*vy-y*vx]$ gives another invariant - sectorial speed $\sigma = \pi R^2\sqrt{(1-\varepsilon)}/T = v_0R/2$.
Keplers's second law: The radius vector **r** from the Sun to a planet sweeps out equal areas in equal times.

3. For planet on orbit with semi-major axis $a=R/(1+\varepsilon)$ and period $\tau=T/(1+\varepsilon)^{3/2}$ we have third invariant $a^3/\tau^2=R^3/T^2$.
Kepler's third law: The squares of the periods of the planets are proportional to the cubes of their mean distances from the Sun.

And most important conclusion - : Simplification of $\sqrt{(ax^2 + ay^2)}*r^2$ gives a constant $4\pi^2R^3/T^2$.
This constant is γM, where γ is a universal constant known as the constant of gravitation and M – the mass of the force center (Sun, Earth, …).
Law of universal gravitation: the falling acceleration due to the force center (of Sun, Earth,…) is inversely proportional to the square of the distance r from the center of force.

Using parametric plotting properties of ***DERIVE*** we can plot $x(t)$, $y(t)$, $r(t)$ and predict the position of the planet at given time t with necessary precision.
After definition of t, x and y in scaled coordinate system (R=1, T=1) simplification of

VECTOR([t,$\sqrt{x^2 + y^2}$)],ε,[0,0.5,0.8,0.9, 0.99])

gives the possibility to plot $r(t)$ for different eccentricity (Fig. 2)

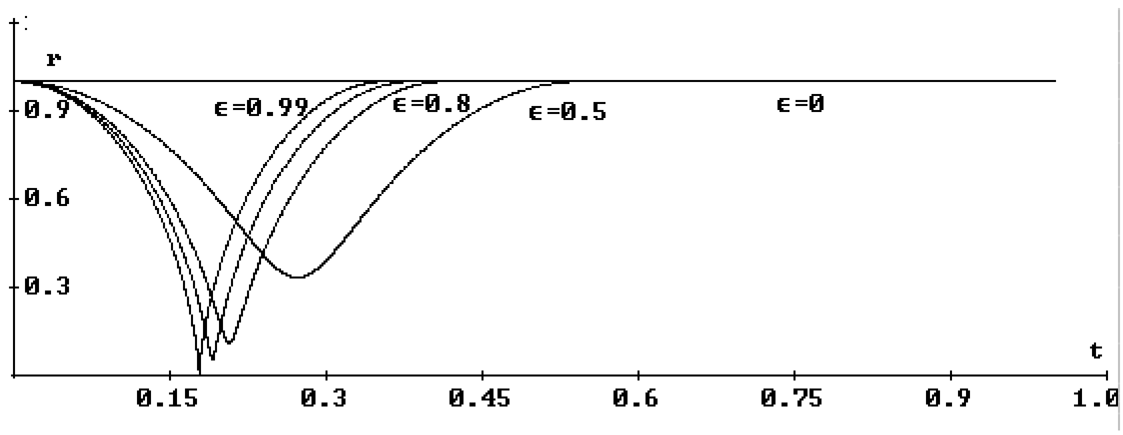

Fig. 2. Time dependence of distance r to force center for different orbits.

The powerful ATAN(y,x) command in ***DERIVE*** enables to calculate and plot time dependence of the angle φ of the radius vector **r**, which characterizes the position of planet in polar coordinate system.

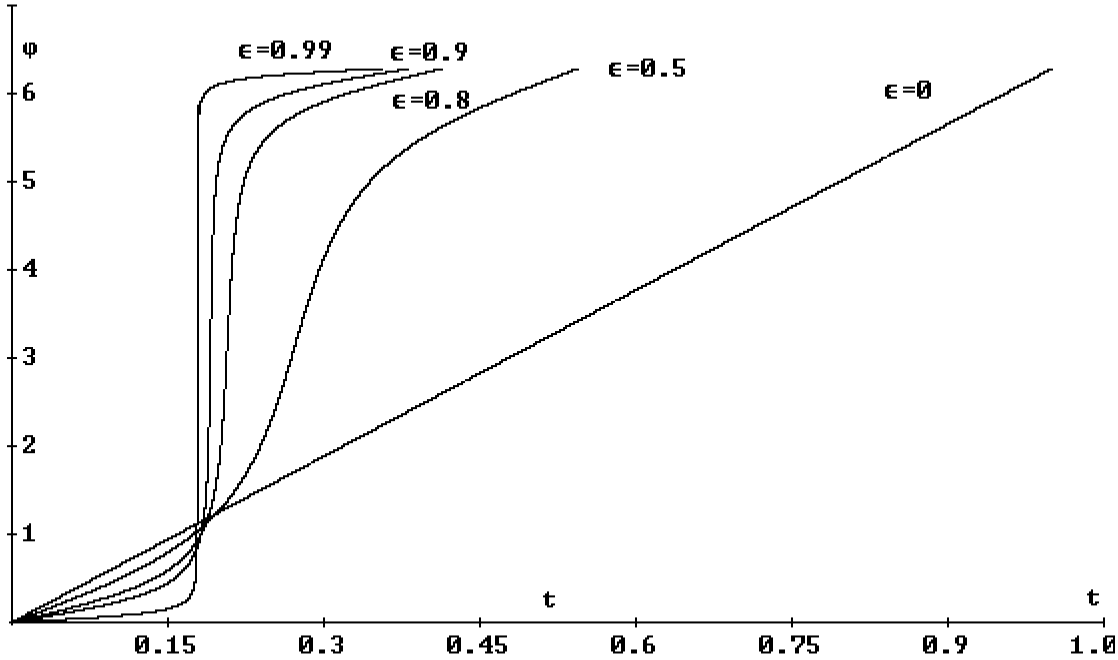

Fig. 3. Time dependence of angle φ of radius vector **r**.

An interesting characteristic of the Kepler's ellipses is that its velocity hodograph is a circle. Plotting [vx,vy] and circle with radius $v_+=v_0/\sqrt{(1-\varepsilon)}$ and origin shifted by $v_-=\varepsilon v_0$. we will find, that hodograph (presentation of motion in velocity space) is this circle. The velocity vector can be calculated very simple with help of two vectors: $\mathbf{v}=\mathbf{v}_+ + \mathbf{v}_-$, which means, that at any moment of time the vector of the planet's velocity **v** is the sum of two independent velocity vectors. The planet moves in a permanent direction (at right angles to the major axis) at constant velocity v_-, and at right angles to the radius vector Sun-planet at constant velocity v_+ [4].
So, the non-solved problem in teaching physics about planetary motion is not a problem in computer algebra like ***DERIVE***.

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References:

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