Squaring the Circle and Leonardo’s Vitruvian man

Today I want to talk about
• A mathematical problem, which is about 3000 years old – squaring the Circle
• A 500 years old famous drawing – Leonardo’s Vitruvian man and
• A classroom experiment using new technological tools – Derive, Cabri, TI92

Two years ago, a friend of mine brought some informations supposing that I was interested.
Reading the paper I found this:
„Geometrical secret of Leonardo da Vinci discovered“
„Leonardo’s solution of squaring the circle“

My first reactions were:
„That’s impossible, we all know, the problem of squaring the circle has no solution!“
„The drawing is 500 years old and nobody discovered this before??“

My feelings wavered between doubting scepticism and „That’s unbelievable, I must check it!“
But my friend was right: I was interested!!
When in 1998 this book was published, I had a lot of informations to work with:
Klaus Schröer (a mathematician) and Klaus Irle (an art historian) worked together and published this book:

Klaus Schröer/Klaus Irle
„Ich aber quadriere den Kreis ...“
Waxmann, Münster, 1998
ISBN 3-89325-555-9

Leonardo da Vinci
First of all I want to make some remarks on Leonardo’s life and work.
He was born in the year 1452 in Vinci, that is a little town near Florence in Italy. On invitation of the King of France, he spent the last three years of his life in Amboise in the valley of Loire. When he was travelling from Italy to France in the year 1516 he had in his luggage three paintings, one of it the famous „Mona Lisa“. That is the reason why nowadays you can see this painting in Paris (Louvre) and not in Italy.

Hundreds of researchers worked on Leonardo’s manuscripts and assets. He is one of the best investigated artist in the world. Nevertheless we still have surprising results.

Art and Mathematics
In the last 500 years nobody analysed the drawing under a mathematical point of view!
What is the reason?
Between the time of Renaissance and today we have a tremendous change in the conception of art. On Leonardo’s point of view, the concept of art is connected with rules, exercises and ingenuity. All artistic subjects can be explained with simple mathematical concepts. Today the development of art is associated with the absence of rational thinking. The artist expresses his feelings.
The view on the mathematical message of this drawing was covered with a smoke-screen in an intellectual climate, in which art and mathematics are mutually exclusive.

Squaring the Circle
Squaring the Circle is one of three great problems of Classical Geometry and in the Renaissance the most famous problem of Mathematics. For more than 3000 years
mathematicians have worked on the problem of constructing a square equal in area to that of a given circle. Whether or not it is possible depends, of course, on what tools you allow yourself. Plato insisted that the problem be solved with straightedge and compass only.

Remark: When Lindemann proved in 1882 that Pi is transcendental he effectively proved that the solution is impossible in a finite number of constructions.

Leonardo expressed several times his intention to write a book about Geometry. In this book he intended to describe various procedures to solve the problem of squaring the circle. But this book was never finished!!

**Vitruvian man**

(Vitruv, Roman architect, wrote 30 B.C. the book „De Architectura“, which influenced architecture in the Renaissance)

In the Internet you can find a lot of Web-Sites with Leonardo`s drawing, but nearly half of the pictures are reversed. Possibly the authors didn´t realize that Leonardo wrote all his texts in mirror-writing!!

In the classroom we made exact measurements and compared this with Leonardo`s text. Comparing areas of the square and the circle (center is the man`s navel) you can easily see a difference.

Circle: 176.72 cm²
Square: 153.51 cm²

Schröer and Irle focussed there eyes on the rotation of the arms. The center of the rotation is marked in the drawing. (Ratio of length of the arms and length of the square is \( f_{ak} = 0.436 \))

First of all they asked for a circle possibly equal in area with the given square. So they realized that the circle you can easily construct has nearly the same area as the square!

**All research workers didn´t pay attention to this circle !!**

Circle: 153.94 cm²

But what`s about a second square equal in area with the larger circle?? (Leonardo`s drawing contains the characteristic of a so called emblem, which were used in the 16th and 17th century)

Schröer and Irle found the missing link:
Connect the corner of the square with the center of the circle and you can construct a larger square (nearly) equal in area to the circle.

Square: 176.89 cm²
The real surprise is the procedure for a construction, defined in this way:

- We start with a square and a circle intersecting the square’s sides.
- Connect upper intersection points, multiply length of the square with 0.436 and we get midpoint of the rotation of the arms.
- Intersection point of arm rotation with upper side of the square is marked.
- Perpendicular bisector yields center point of the larger circle.
- Construct the larger circle.
- Connecting the edge of the square and center point of the larger circle and you can draw intersection point with the larger circle.
- Second square is drawn.
First Calculations

In the classroom we started with an example:
\( \text{a1} = 10 \) and \( r1 = 5.4 \)

(Geometrical construction and algebraic calculations parallel)

- For the geometrical constructions (by hand) we used straightedge and compass. I think, it is necessary to use the fingers for at least one example!!
- For the calculations we used Derive.

You need equation of a straight line and equation of a circle in this form:
\[
(x - x_m)^2 + (y - y_m)^2 = r^2
\]

The role of the CAS:
We use Derive to solve 4 equations but the process must be **controlled by the students**.

Questions

- Does the sequence of ratios of areas (circle – square) converge?
- What’s the limit of this sequence?
- What’s the effect of other dimensions of the initial pair (circle – square)?
- Does the sequence converge, if you use different factors \( \text{fak} \neq 0.436 \)?
- What’s the effect of the factor on the limit of the sequence?
Geometrical Iteration with Cabri

One of the most powerful tools is the application of macros. We use for simplification \( fak := 0.5 \)

Iterations with Derive

We cannot be satisfied with these results. Our next calculation (algebraic-analytical methods) uses the ITERATES-Function of Derive.

Syntax of the \textbf{ITERATES}-Function:

\begin{center}
\texttt{ITERATES( Operation , Variable , Initial Object , Number of Iterations )}
\end{center}

Informations in a row:
\begin{itemize}
  \item Length of initial square \texttt{a1}
  \item Radius of initial circle \texttt{r1}
  \item (Calculated) length of final square \texttt{a2}
  \item (Calculated) Radius of final circle \texttt{r2}
  \item Area of final circle
  \item Area of final square
  \item \textbf{Ratio} of this areas in percent
\end{itemize}
Operation (generating a row by the preceding row):

Take the 3\textsuperscript{rd} and 4\textsuperscript{th} value in the following row as initial objects!!

\[ \text{ITERATES} \begin{bmatrix} a_1 := z_3, r_1 := z_4, a_2, r_2, w \cdot r_2, a_2, \frac{w \cdot r_2^2}{a_2^2} \cdot 100 \end{bmatrix} = [z, \text{startzeile}, 10] \]

\[ \text{startzeile} := \begin{bmatrix} 0, 0, 10, 5.4, 5.4, 10, \frac{5.4^2}{10^2} \cdot 100 \end{bmatrix} \]


**Different initial objects:**

\[ \text{fak} := 0.436 \]

\[ \begin{array}{c|c|c|c|c|c|c} a_1 := 10 & r_1 := 5.2 & r_1 := 5.6 & r_1 := 6.0 & r_1 := 8.0 \\ \hline 84.9486 & 99.5203 & 113.097 & 201.061 & \\ 103.204 & 100.204 & 99.2032 & 106.642 & \\ 99.7444 & 100.019 & 100.127 & 99.5 & \\ 100.068 & 100.039 & 100.027 & 100.094 & \\ 100.034 & 100.037 & 100.038 & 100.031 & \\ 100.037 & 100.037 & 100.037 & 100.037 & \end{array} \]

**Different factors:**

\[ \text{fak} := 0.35, 0.45, 0.46, 0.5 \]

\[ \begin{array}{c|c|c|c|c|c} 84.9486 & 84.9486 & 84.9486 & 84.9486 & 84.9486 \\ 105.305 & 103.021 & 102.902 & 102.500 & \\ 99.7254 & 99.7364 & 99.7301 & 99.7029 & \\ 100.351 & 100.034 & 100.012 & 99.9334 & \\ 100.265 & 100.004 & 99.9845 & 99.9126 & \\ 100.277 & 100.007 & 99.9873 & 99.9144 & \\ 100.275 & 99.9871 & 99.9870 & & \end{array} \]
Summing up

The special attraction is to work at different levels:
- Historical aspects
- Geometric-constructiv work with compass and straightedge
- Calculations of intersection-points (line – circle), lengths and ratios
- Constructions with a DGS
- Iterations with a CAS

The real fascination is not answering the question for the problem of squaring the circle, but the convergence of that sequence generated by Leonardo´s procedure!!

That's the real secret of this drawing!!

Student’s reactions

„Most important for me was, that mathematics is not only a boring manipulation of terms, but can be an exciting puzzle.“
„I liked the variation“
„Most important for me was to be able to control the computer-applications for the solutions“
„I liked to discover a little bit of Math-history“
„The actuality!“