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New emphasis of fundamental algebraic competence and its influence in exam situation

1. The task of schools in the age of information technology and the role of mathematics

1.1 Education versus Qualification

The discussion concerning the duty of the school of the future is characterized by a polarisation:

- On the one hand we have the central necessity of a more global and longer term education. When talking about education, we refer to both the process of educating and the resulting product, namely education. In other words education is the acquisition of cultural resources such as languages, art, knowledge and techniques and, at the same time, the personality formed in this process, the autonomy of the individual in contrast to the collective, historically changeable demands. In the center, forming the personality is the confrontation with certain ideas which are not determined by current short term social or economic demands, but rather defined by an independent conception of education.

- On the other hand, we expect schools to provide qualifications, qualifications which are a sum of necessary skills, abilities and areas of knowledge which are vital to carry out certain jobs. The person as an independent personality in his general role in society remains, for the most part, unheeded. Such a training is determined by the production of job qualifications.

Thesis1:

The school of tomorrow in an effort to prepare pupils for life long learning must be able to do both: educate our youth and equip them with the necessary qualifications

Therefore, educators must not limit themselves to simply providing subject competence. In order to fulfill the mission of education it is necessary for teachers to equip the students with various **key qualifications** [Klippert, 1999]

- **subject competence**
- **methodological competence**
- **social competence**
- **personal competence**

Subject competence:

This year´s conference deals with the fundamental question of mathematical competence, so it goes without saying that I will discuss this subject competence at a later point in my lecture

Methodological competence:

Independent accumulation of information, productive usage of information, skilled usuage of heuristic strategies, choice of appropriate media, adequate presentation techniques, systematical practice and repetition

Social competence;

Communication and co operation skills

Personal competence:

Independence, self confidence, self evaluation, motivation, willingness to perform, language competence, logical thought process

1.2 Ability versus competence

Another point of view which is important for the ideas of my lecture comes from a discussion with B. Kutzler about the difference beween the concepts of **„competence“ and „ability“**

- The **ability** to do something is the quality or skill in doing a particular thing
- Having the **competence** includes more than just being able to do something. It means doing something with understanding and doing it based on a personal decision and because of one´s own personal considerations. Furthermore, competence implies that something is done well.

Thesis 2:

In the school of tomorrow students should not only acquire abilities, they should gain personal competence

1.3 Short term – versus long term competence

Information technology has made it possible through statistical tests to question and evaluate school quality. We need only to think of the much discussed TIMS study. However, a common mistake which is always made in the interpretation of the results of such studies is the following:

There is no difference made in:

- The **short term competence** of having skills and knowledge readily available for a certain learning process and
- The **long term competence** of being able to recall and retrieve knowledge or ability much later to solve an a current problem

If we are interested in evaluating the quality of education we need to pay close attention to the evaluation of the competence of recalling at a later moment and not testing only what the student knows immediately following a certain learning process.

This temporarily available quality must be more detailed and more extensive than the long term quality of recalling and retrieving. In this way, I understand the thesis of our article ***„Indispensable Manual Calculation Skills in a CAS Environment“*** which W. Herget spoke about in his key note. The manual calculation skills which we demand in this article are typical long term recalling/retrieving skills. The abilities in the actual algebra learning phase must be more extensive.

Thesis 3:

The short term competence, which should especially be available for a certain learning process, must be more detailed and more extensive than the long term competence of recalling and retrieving.

1.4 The role of mathematics

To fulfill the task of schools in the age of information technology mathematics is of fundamental necessity.

We live in a mathematical world. Whenever we decide on a purchase, choose an insurance or health plan, or use a spreadsheet, we rely on mathematical understanding. The World Wide Web, CD-Roms, and other media disseminate vast quantities of extensive information. The level of mathematical thinking and problem solving needed in the workplace has increased dramatically.
In such a world , those who understand and can do mathematics will have opportunities that others do not. Mathematical competence opens doors to productive futures . A lack of mathematical competence closes these doors.

But the vision of mathematics teaching and learning is not the reality in the majority of classrooms and schools. In this age of information technology students need to learn a new set of mathematics basics that enables them to solve problems creatively and resourcefully.

The National Council of Teachers of Mathematics in USA (NCTM) formulated guidelines for educational decisions laid out in principles and standards .
One of these principles is called the „Technology Principle“:

> Calculators and computers are reshaping the mathematical landscape, and school mathematics should reflect these changes . Students can learn more mathematics more deeply with appropriate and responsible use of technology.

The investigations of the Austrian CAS projects justify an optimistic point of view. The major results are not especially mathematical contents - it is a more pupil centered, experimental way of learning. In other words, the new tool supports all of the four key qualifications which I mentioned at the beginning.

Thesis 4:

CAS-supported mathematical education supports and encourages the 4 key qualifications

- **Subject competence**
- **Methodological competence**
- **Social competence**
- **Personal competence**

much better than traditional mathematics education.

Referring to the mathematical competence (subject competence) the use of technology has changed the needs of fundamental mathematical competences and skills which are the basis of the central task of the subject mathematics for problem solving.

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In Austria we are now thinking about a new curriculum which, on the one hand, formulates the necessary fundamental competences and, on the other hand, expresses the contribution of mathematics for gaining the important key qualifications especially the qualification of solving problems by using the mathematical language. But we also have to start a discussion about standards which express the pupils´ expected behaviour especially in exam situation.

I have chosen one fundamental competence which has significantly changed because of the use of CAS

2. The necessary fundamental algebraic competence in the age of CAS

In the same manner as we discussed the necessary numeric fundamental competence when we introduced the numeric calculator in the Seventies, it now becomes vital to explore the algebraic fundamental competence . This topic will deal with these questions partly only by formulating new questions but I will also try to give first answers based on the experience of the Austrian CAS projects.

2.1 The situation at the beginning of the CAS-age

> *In the Middle Ages there existed a honorable guild, the guild of "calculating masters". They died out when people became able to calculate themselves. Will the math teachers also die out in the age of CAS?*
>
> *The progress of humanity is documented by her tools . Tools, on the one hand, are results of recognition and on the other hand new recognition is not possible without tools. [Claus, 1990, p 43]*

In the 17th century Leibnitz tried to invent a calculating machine because *"for human beings it is unworthy like slaves to lose hours by doing stupid and monotonous calculations".*

But looking at math lessons or tests, calculating skills are still dominant.

In the didactic literature one can find dubious arguments, like [Köhler, 1995]:

Calculating

- supplies math education with necessary materials for practicing
- makes a feeling of success possible also for the weaker students
- relieves teachers and students.

Or another argument which I found in an didactic article:
Technological progress causes the disappearance of routine exercises which brought a feeling of success for weaker students by practicing hard.

My opinion is: It is senseless to hold on to senseless tasks or goals because they give weaker students a feeling of success. This is a danger for the subject of mathematics because people would look for more sense in other subjects as you can see in articles in Austrian newspapers titled “Environment instead of mathematics”.

Thesis 5:

> **In the CAS-supported math education it is also possible or even more easily possible to find routine exercises suitable for more meaningful goals and to automate skills which give weaker students a feeling of success.**

He who holds on to the dominating of calculating skills should look at the history of mathematics. A main goal of mathematics always was to develop schemes and algorithms which make lengthy calculations obvious, that means to automate unnecessary mathematical activities or in other words, to trivialize. Progress in mathematics means creating free capacities by automatism for more important activities on a higher level. Tools always play an important role in this process..

As the results of our CAS projects and the developed didactic concepts show, mathematics in the age of CAS does not necessarily mean uncomprehending work with black boxes. Especially the White Box/Black Box principle demands at first a white box phase in which the algorithm is developed also trying to understand the process. and in which the learners have to acquire certain fundamental manual calculating skills. Only afterwards operating is done by the CAS as a black box. The CAS plays an important role in both phases: In the white box phase as a didactic tool to help the learners to make the box “white” and also as a controlling tool , in the black box phase as a calculating tool and as a tool for testing and interpreting.

But the theme of my lecture is not only manual calculating competence. Fundamental algebraic competence is more than calculating, but the main change is the change of the importance of calculation competence.

The tool computer forces us to think about problems which we had to think over a long time ago [Schubert, 1994].

2.2 Important elements of fundamental algebraic competence

We first have to ask ourselves what the educational value and the goals of the subject mathematics are and then consider the question of partial algebraic competence. As this discussion would go beyond the scope of this conference, I have based my thoughts concerning the grander picture of mathematics on the following definition by Bruno Buchberger

> ***Mathematics is the technique, refined throughout the centuries, of problem solving by reasoning***

Let us now turn to those competences which, in my opinion, are important:

2.2.1 The competence of finding terms or formulas

Of the three phases of the problem solving process, modelling—operating—interpreting, the operating phase has always dominated. The tools of CAS now make it possible to more evenly distribute the importance of the three phases. This means that developing formulas gains more importance in comparison to calculating with terms.

The activity often is a brain activity by remembering an earlier learned formula or knowing where the needed formula could be found.
This competence also demands abilities of other areas of mathematic activity like competence in geometry for deriving geometric formulas or the important competence of translating a text into an abstract formula.

The translation process from the student´s language into the language of mathematics mostly takes place in three steps:
Step 1: A sentence in the colloquial language
Step 2: A "word formula"
Step 3: The symbolic object of the mathematical language

Didactic concepts which support the acquisition of this competence could be taken from language subjects, e.g. creating a dictionary for translating from German into mathematics.
Example: The sentence "*vermehre um p Prozent*" is translated into "*.(1+p/100)*"

The influence of CAS:

- The CAS allows the students to transform the word formula directly into a symbolic object of the mathematical language by defining variables, terms or functions or writing programs.
- The CAS allows a greater variety of prototypes of a formula and also offers some which were not available before. While in traditional mathematics education often only one prototype is available and used, now the CAS offers several prototypes parallely in several windows. This fact causes a new quality of mathematical thinking. We name this mathematical activity the "Window Shuttle Method"
- The CAS offers and allows a greater variety of testing strategies, in this case testing if the formula is suitable for the problem and mathematically correct.

Example 2.1: A financial problem

> One takes out a loan of K=$ 100.000,- and pays in yearly installments of R=$ 15.000,-. The rate of interest is p=9%. After how many years has he paid off his debts?

In traditional mathematics education such problems could be solved for the first time in 10^{th} grade, because the students need geometric series and calculating skills with logarithms. The computer offers a new model, the recursive model. From that pupils now work such problems in 7^{th} grade

The first step is finding a word formula, which describes what happens every year:

> *Interest is charged on the principal K, the instalment R is deducted.*

Translated into the language of mathematics:

$$K_{new} = K_{old}*(1+p/100) - R$$

After finding a formula in the phase of modelling operating is the next activity. Using a CAS calculating takes on a new meaning. The TI-92 offers a special way to come to a better understanding of a recursive (or better an iterative) process: The activities of storing and recalling make the pupils conscious of the two important steps of a recursive process: Processing the function and feedback ($K_{new} \rightarrow K_{old}$) (figure 2.1). Looking at the list of values the quality of an exponential growth becomes much clearer than by calculating with logarithms. The typical problem of paying in installments can be recognized: During the first phase the loan is nearly equal because the greatest part of the installment is used for the interest (figure 2.2). The experimental solution is obtained by repeated usage of the enter-key until the first negative value appears (figure 2.3). The variable n shows the number of the years.

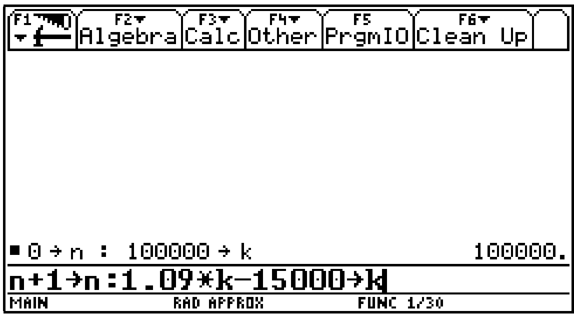

figure 2.1

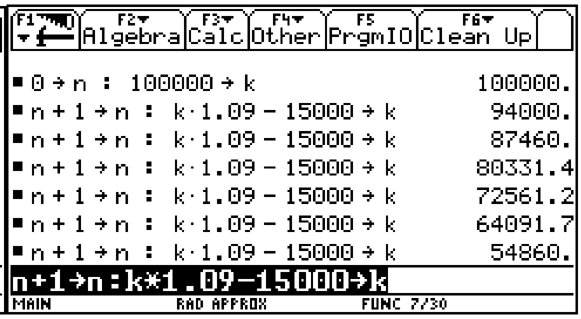

Figure 2.2

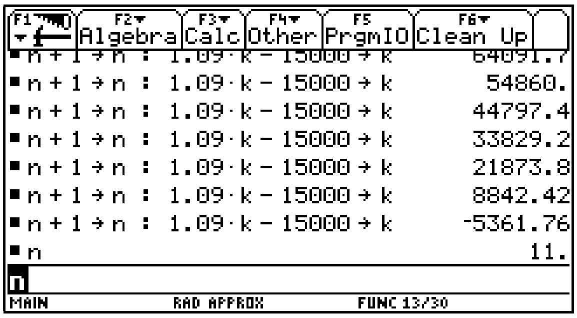

figure 2.3

2.2.2 The competence of recognizing structures and recognizing equivalence of terms

This competence is necessary when developing a term, when deciding upon or entering a certain operation and also when interpreting or testing. This competence has always been of great importance as research, such as that of Günter Malle, has shown us that the most commonly made mistakes during algebraic operating are those of recognizing structures.

Recognizing equivalence of terms which the learners have developed or recognizing results of calculations done by the CAS is a part of the competence of recognizing structures.

A prerequisite for this competence is the knowledge of basic algebraic laws. Such decisions cannot be done successfully by using the CAS as a black box without this mathematical knowledge.

The influence of CAS:

- When using a CAS the first step, the input of an expression, needs a structure recognition activity.
- Using the CAS as a black box for calculating a recognition of the structure of the expression is necessary before entering the suitable command. Blind usage of commands like *factor* or *expand* is mostly not successful.
- The learner must interpret results and recognize their structure which he himself did not produce.
- The individual results of various students doing experimental learning must often be checked for their equivalence.
- CAS sometimes produce unexpected results and students do not know whether they are equivalent to their expected results or whether they differ.

Example 2.2: New goals for working with traditional complex expressions [Böhm, 1999]

In traditional mathematical school books one can find such complex terms for practicing manual calculations. Using a CAS like the TI-92 or Derive the new goal is structure recognition when entering the expression (figure 2.13 And 2.14). Especially the linear entry line demands this competence.
The calculation is done by the CAS as a black box (figure 2.14 And 2.15). We think that the competence of manual calculating such a complex expression necessary in the age of CAS.

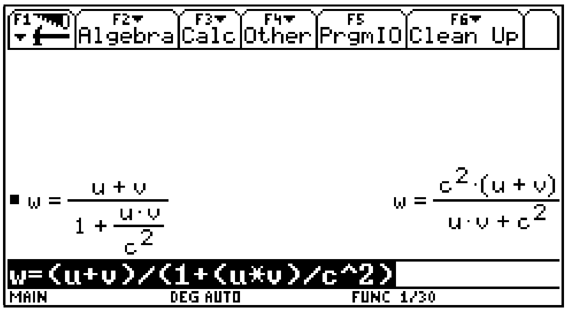

figure 2.13

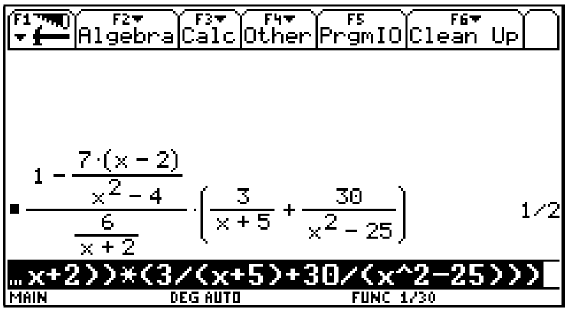

figure 2.14

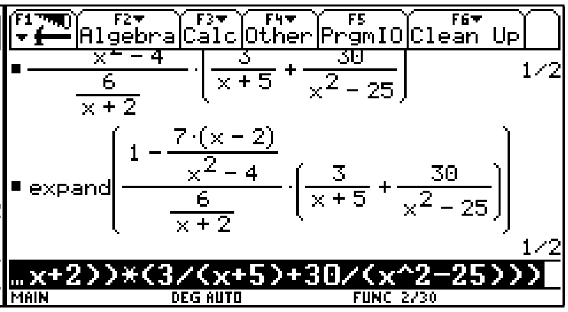

figure 2.15

Example 2.3: Riemann sums

Calculate the definite integral $\int x^2 dx$ taken from a to b by using the definition of the definite integral. Compare three ways: „midsums“, „lower sums“ and „upper sums“. Make at first a sketch and test the result with the formula of the definite integral for a= -2 and b=4. Use also the TI-92 as a black box for testing (12^{th} grade)

Way 1: Midsums

The expected result is $b^3/3$ - $a^3/3$ but the CAS offers a quite different result as you can see in figure 2.26. The best way to come to the expected result is a structure recognition which leads to the decision to use the *expand*-command

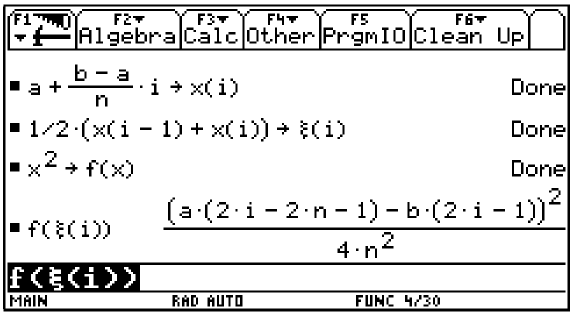

figure 2.24

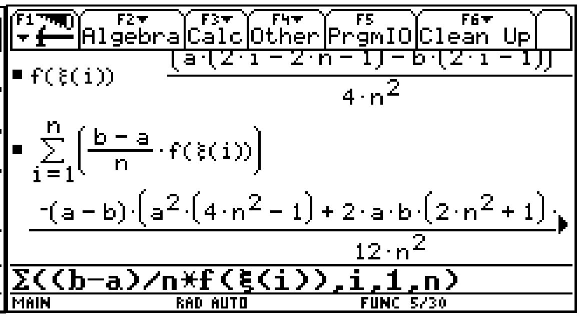

figure 2.25

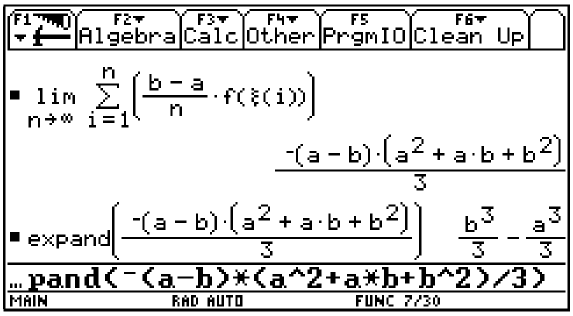

figure 2.26

2.2.3 The competence of testing

Ever since math has been used as a problem solving technique, it has been necessary to corroborate the correctness of the solutions and to interpret them. The teacher has to offer the learners testing strategies or make them able to find some themselves.

In traditional mathematics education testing means activities like substituting numbers, trying another way of solution, checking the usefulness of the mathematical solution for the applied problem, remembering the definition of a concept a.s.o.

A central result of our CAS projects is a more experimental and independent learning process, whereby the expert is not so much the teacher as the CAS. This means that testing becomes even more in important. The stronger emphasis on modelling and interpreting also demands a higher competence in testing.

The influence of CAS:

- The CAS enables the learner to carry out tests both more effectively and quickly.
- Completely new possibilities are available as far as algebraic and graphic testing are concerned
- Using CAS causes a new problem: The learner has to examine and to interpret results which he himself did not produce. The expectation of the sort of the solution or the form of the algebraic term sometimes differs between the learner and the machine.

- The variety of paths leading to solutions and therefore the number of different results increase dramatically. One will not often find the “algorithmic obedience” of the classical math classroom, in which the majority of the students simply imitate the strategies presented by the teacher. Therefore the equivalence of the numerous results has to be tested.
- The more applied mathematics which we see in the CAS-classrooms demands more testing of the correctness of the model, testing of the usefulness of the mathematical solution according to the given problem and testing of the influence of parameters.

Due to the growing importance of testing , new strategies are necessary which show the learner how to make use of the potential possibilities of the CAS. All those who fear that the use of the computer will lead the learner to experimenting with black boxes without the faintest comprehension of what he is doing should realize that in order to carry out an activity on the computer, the learner must, very definitely, have a grasp for algebra and the underlying algorithm. In fact he needs a wider comprehension than if he were to do the problem by hand. A coincidental trial and error method would not be successful.

Example 2.4: Interest is paid on the capital k at the percentage rate p (7th grade)

Determine a formula for the new capital after one year

Pupils found several formulas, some of the results were wrong (figure 2.27 and figure 2.28)
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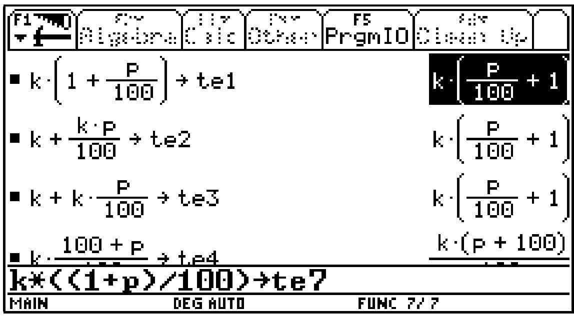


figure 2.27

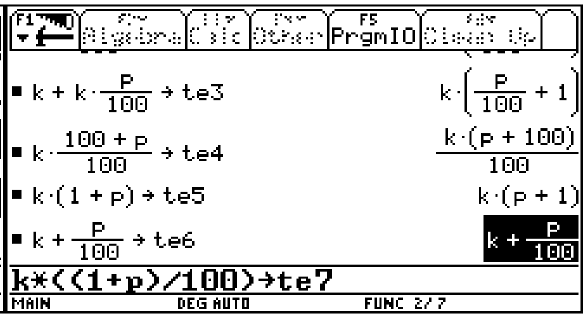


figure 2.28

Now it was necessary to find strategies to prove the correctness and the equivalence of the terms.

Strategy 1: Using the algebraic competence of the TI-92

After entering an expression in the Entry Line, the TI-92 produces entry/answer pairs. The answer is the simplified version of the entry term, normally simplified by factorizing. When the pupils compare the factorized answer terms they find out, that the terms te1, te2 and te3 are equivalent. They were not sure whether te4 is also equivalent, but it is rather sure that te5 and te6 are not equivalent to te1, te2, te3 (figure 2.28).

Strategy 2: Using the difference of the terms

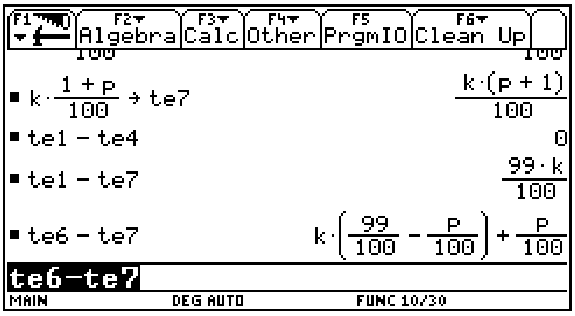


figure 2.29

If the result is 0, pupils recognize that the terms are equivalent. Building the difference of te1 and te7 one can see, the terms are not equivalent.

Strategy 3: Using equations

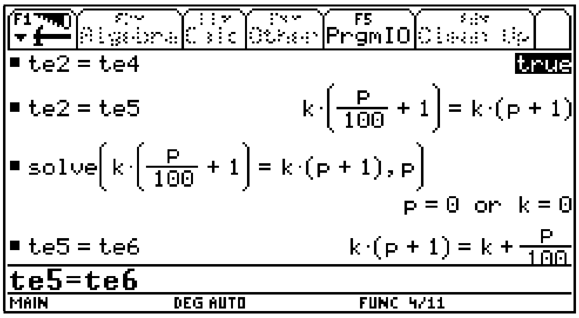


figure 2.30

If the answer is „true“, the equation is soluble for all numbers of the domain. So the pupils know the terms are equivalent. In all other cases the equation is only soluble for certain values and therefore the terms are not equivalent.

Strategy 4: Looking for a factor

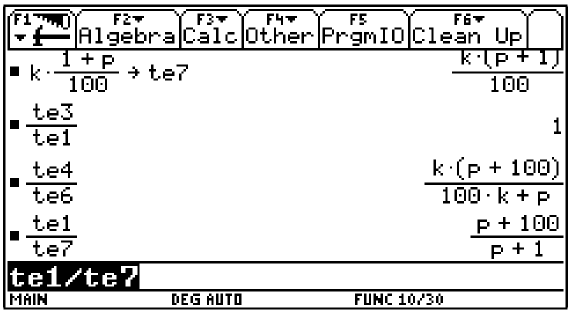


figure 2.31

If the quotient of two terms is 1 they are equivalent.

Strategy 5: Substituting numbers

This traditional strategy can now be used more easily because calculation is done by the CAS

2.2.4 The competence of calculating

Before discussing which sort and what extent of competence is necessary for the students to have we might first define what calculation competence is:

Definition: Calculation competence is the ability of a human being to apply a given calculus in a concrete situation purposefully.

This definition shows that calculation competence does not only mean to execute a certain operation. Most important for us is the distinction between the goals “perform an operation” (to some extent this can be delegated to a calculator) and “choose a strategy” (this cannot be done by the calculator.)

Manual calculating skills are a branch of the calculation competence, because having calculation competence could also mean being able to decide on the suitable algorithm and to delegate the execution to the computer. But we are still dedicated of the following thesis

Thesis 6:

For mathematics to develop within a learner certain calculation skills are needed.

We cannot completely leave the calculations to the computer as a black box.

When I say “we” I also mean the group Herget, Lehmann, Kutzler Heugl. We expressed our position into the paper entitled

Indispensable Manual Calculation Skills in a CAS Environment

which Wilfried Herget spoke about yesterday.

In one topic my position was different from the others. I demanded to distinguish between short term and long term competence. And my position was that

> **the calculation short term competence, which should especially be available for a certain learning process (the white box phase of elementary algebra), must be more detailed and more extensive than the long term competence of recalling and retrieving.**

We teach manual calculation skills not only for their own sake, I am convinced that they are prerequisites for the attainment of most of the fundamental algebraic competences which I speak about in this lecture.

I will go more detailed into this aspect in chapter three where I will speak about the exam situation.

Another point of view:

Richard Skemp [Skemp, 1976] distinguishes between **relational understanding** and **instrumental understanding** (or shortly between understanding and skills):

Instrumental understanding: Mathematical usage of rules when solving problems without necessarily knowing why the rule is valid.

Relational understanding: The ability of deriving rules, interpreting and possibly proving, to see them as rules in a net of concepts ("knowing both, how to do and why).

This point of view leads to some questions:

Question 1: Is instrumental understanding a prerequisite or a support for a higher level of relational understanding?

Question 2: Does relational understanding support the necessary skills of instrumental understanding?

An investigation of our last Austrian CAS project gave us the following answers: [Heugl, Gösing 1999]

Answer to question 1:

Instrumental understanding is not an absolutely necessary prerequisite for a higher relational understanding. The ability to give reasons does not automatically go hand in hand with the ability to calculate.

Answer to question 2:

Relational understanding does not necessary support the skills of instrumental understanding. Or in other words: To have relational understanding is not enough for having instrumental skills.

I admit that the answer to question 1 is a certain contradiction to the former formulated thesis that manual calculation skills are a necessary prerequisite for the attainment of most of the fundamental algebraic competences.

Our research projects are just giving not only answers they are also producing open questions. We need didactical discussions and meetings like this congress to come to clearer positions.

The influence of CAS in the calculation competence:

- A shift in emphasis from calculating skills to more conceptual understanding, to more modelling and interpreting.
- A shift from doing to planning.
- A reduction of the complexity of manual calculated expressions.
- A shift from calculation competence to other algebraic competences, like structure recognition competence or testing competence.
- A better connection between the formal aspect of mathematics and the aspect of contents.

2.2.5 The competence of visualizing

A special quality of mathematics is the possibility of graphic representation of abstract facts. Visualizing was also important in traditional mathematics education but it was not easy to get the graphic prototype of a concept or a function. Apart from free hand drawings, it is difficult to develop graphs without using a computer. Finding the most important points and characteristics of functions in order to be able to draw the graph is the main goal of discussion of curves in analysis.

The influence of CAS:

- CAS allows the learner to get the graph faster and more directly.
- Other prototypes of a concept or specially a function are also available much more easily, like a table or lists of values or matrices in a Data/Matrix editor.
- The CAS allows the learner to use several prototypes parallely, while in traditional math education only one prototype was given e.g. the term and it was hard work to get other prototypes like the graph or a table. Now the given term allows the pupil to draw the graph directly by entering the command *graph* and deciding on a suitable window area.
- The learning process consists of shuttling between several prototypes that means shuttling between several windows. Therefore we call this didactical concept the Window Shuttle Method.
- These facts also allows to solve algebraic problems graphically.

Example 2.5: Graphic solution of an unequation. Solve the equation

$$|x - 2| - 1 < x/3 + 1$$

The *solve*-command does not help the learner to find a solution (figure 2.39). Graphic solution means to comprehend the left and the right side of the unequation as functions (figure 2.40), to draw them and to look for the intersections in the graphic window (figure 2.41). These can be found by calculating when using the CAS as a black box or by experimenting or walking along the graph by using the Trace-mode.

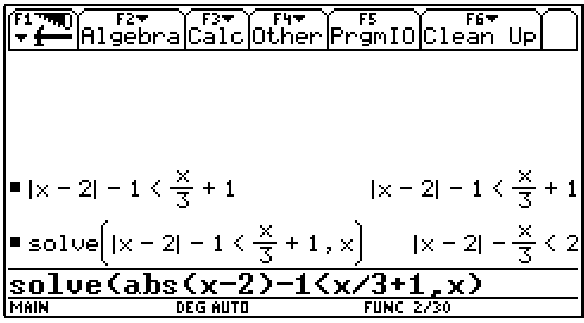


figure 2.39

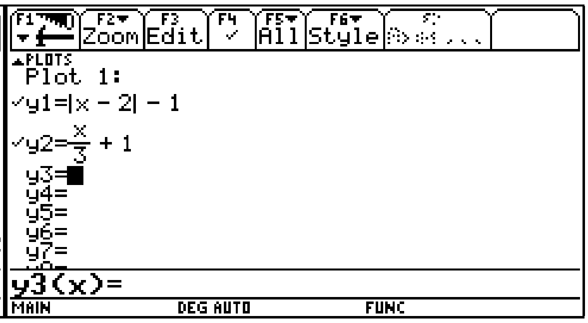


figure 2.40

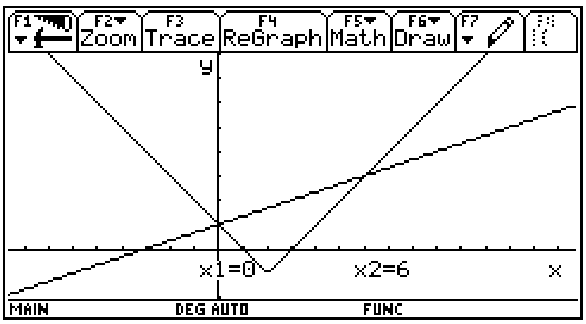


figure 2.41

2.2.6 The competence of working with modules

Using modules is not new for the learners. Every formula used by the pupils can be seen as a module e.g. Hero's formula for the area of a triangle or the use of the cosine rule in trigonometry. Knowing such a module means that the student has a model for his problem but when using this module he has to do the calculation himself.

The influence of CAS:

- The computer, and especially CAS, opens a new dimension of modular thinking and working. By defining or storing parts of a complex expression as a variable, students can simplify the structure of the expression making it more comprehensible and they can calculate with such modules.
- One weak point of the TI-89 and TI-92 promotes and strengthens the use of modules as new language elements: the small screen. More complex expressions cannot be totally seen on the screen and therefore operations with such expressions become confusing. So the structure of such operations is more comprehensible and clearer if students use the name of the expressions instead of the expressions themselves.

Example 2.10: Solving systems of equations. Discovering algorithms like substitution method, Gauss algorithm a.s.o.

Watching our students from seventh to tenth grade in the White Box phase of learning we observed three steps of abstraction:

Step 1: Working "***into the equations***" as in traditional math education.
Step 2: Working "***with the equations***".
Step 3: Working "***with the names of the equations***", that mean working with modules.

Step 1: In traditional math education students have to work ***„into the equations"***. They have to do the calculating themselves:

(I) 3.x - 2.y = 12 |+2.y
(II) 7.x + 2.y = 8

(I) 3.x = 12 + 2.y |:3
(II) 7.x + 2.y = 8

(I) x = (12 + 2.y)/3
(II) 7.(12 + 2.y)/3 + 2.y = 8 |·3

(II) 84 + 14.y + 6.y = 24 |-84
(II) 20.y = -60 |:20
(II) y = -3
a.s.o.

Step 2: Using CAS students can work ***„with the equations“***. Calculating, substituting and using algorithms to solve the single linear equations, which they learned in former White Box phases, is now done by the computer as a Black Box. Students have to decide on the operations, the CAS have to do them (figure2.42).

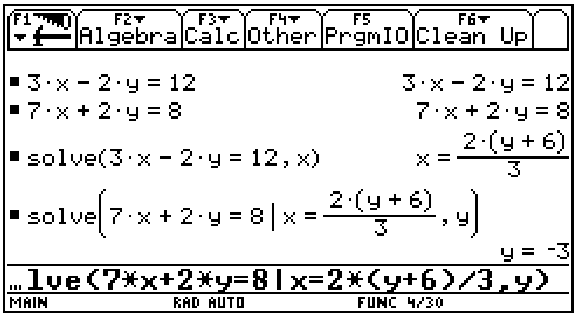


figure 2.42

Linear systems of equations in ninth grade

Step 3: In our project where we observe students who are familiar with the CAS we found out that the operations are not carried out with the equations but ***„with the names of the equations“*** which the students had defined as new elements of the mathematical language.

Using the idea of the Gauss algorithm to find out the Cramer rule.

After storing the equations as named variables students can use the names to form suitable expressions. The next step is to speak about strategies in their colloquial language and to translate it into the mathematical language. E.g.: „*We have to multiply the first equation with a_{22} and the second equation with $-a_{12}$ and have to add the two equations. After that we have to solve the new equation for the variable x_1 and so on.*“

At first for a better understanding of the several calculating steps it would be better to separate the particular steps:

Word formula	Abstract expression
Multiply the first equation with a_{22} and the second equation with $-a_{12}$ and add the two equations.	**a_{22}.equ1 + ($-a_{12}$).equ2**
Store the new equation with respect to the variable x_1 in the variable equ2n	**a_{22}.equ1 + ($-a_{12}$).equ2→equ2n**
Solve the new equation for the variable x_1	**solve(equ2n, x_1)**

figure 2.43

Students having more experience with using CAS and especially the more talented students more and more prefer to translate the word formula of phase 1 into one abstract expression:

Multiply the first equation with a_{22} and the second equation with $-a_{12}$ and add the two equations. Now solve the new equation for the variable x_1	**solve(a_{22}.equ1 - a_{12}.equ2, x_1)**

figure 2.44

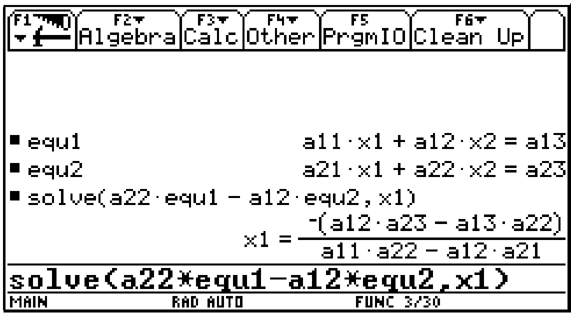


figure 2.5

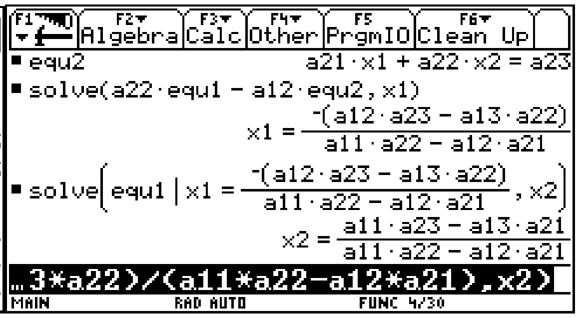


figure 2.46

This result shows a new quality of mathematical thinking caused by CAS (figure 2.44). The tool of CAS does not only support cognition, it becomes part of cognition.

2.2.7 The competence of using the chosen CAS

> *Besides the actual mathematic contents I now have to additionally concentrate on skills which are necessary when using the calculator.*

This statement of one of our students shows the necessity of the “computer-using-competence”.

The influence of CAS:

- The use of CAS causes additional demands and problems for the students. The operation of the electronic tool needs additional skills which also have to be practiced as calculation skills.
- The evaluation of our last project shows that the measured growing joy and interest in mathematics is significantly higher by those pupils who have no problems with the operation of the computer.
- Another significant result is the gap between boys and girls. Both groups show a growing joy and interest in mathematics but boys significantly more than girls. Girls more often observe to have problems with the operation of the calculator.
- The necessary commands, operations and modes have to be offered to the students in small portions. Practicing and repeating in regular intervals are necessary.
- The use of CAS as a Black Box for problem solving demands an agreed documentation of the way of solution, especially in the exam situation.

Some handling skills which are necessary for the fundamental algebraic competence when using a TI-92:

- Input → recognizing the structure of the expression.
- Storing and recalling variable values.
- Most important commands in the algebra menu are *factor*, *expand* and *solve*.
- Substituting numbers, variables and expressions.
- Setting modes which are necessary for algebra, like the decision exact/approx.
- Defining functions for graphing, displaying Window Variables in the Window Editor. Using Zoom and Trace to explore the graph.
- Generating and exploring a Table, Setting Up the Table Parameters.

3. The influence of the use of the tool CAS in the exam situation

Which is the more valid question ?

Do the new ways of mathematics learning and teaching influence the exam situation?
or
Does the exam situation influence new ways of learning and teaching?

In the past the exam situation has always had a great influence on the content and the didactic concept of mathematics education. So the emphasis sometimes placed on a specific math topic can only be explained because it is easy to construct a suitable test.

Therefore it is comprehensible that one of the principles formulated by the American NCTM is called the "**Assessment Principle**"

> *Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.*

Some tasks of the Assessment Principle:

- Assessment should be more than merely a test at the end of instruction to gauge learning.
- Teachers should be continually gathering information about their students.
- Assessment should focus on understanding as well as procedural skills.
- Assessment should be done in multiple ways, and teachers should look for a convergence of evidence from different sources.
- Teachers must ensure that all students are given the opportunity to demonstrate their mathematical learning.

Already in our former CAS projects we recognized that the way of assessment was not suitable to the new ways of learning which we observed in our CAS classes.

In traditional mathematics education written exams (5 or 6 one-hour-tests per year) dominate. As far as content is concerned the emphasis is on calculation skills. This way of testing is suitable to the dominating style of teacher centered teaching, which causes a more reproductive way of learning. Often in two or three of the four examples of a test the same skills are tested – students have to be busy for one hour.

Some significant changes of the learning process in our CAS-classes which strengthen the necessity of changes in the exam situation:

- A more pupil-oriented learning process. More frequently mathematical discussions among the students. The teacher is not the source of knowledge, he supports the independent acquisition of knowledge by the students.
- Experimenting, the trial- and error method: We seldom find the "algorithmic obedience" where the teacher shows one way which all the students then accept and follow
- Working in pairs or groups can be seen much more frequently.
- Beside the teacher there exits now a new, very competent expert – the tool CAS. That means pupils do not always need the teacher for examining the correctness of their ideas and results.
- Phases of "open learning" (nancy check) where the students are individually organizing the velocity and the contents of their learning process.

- The CAS is not only a calculation tool, students can also store knowledge by defining modules or using the text editor. Therefore it is senseless to forbid the use of learning media, like books or exercise books during the tests.
- New emphasis on fundamental competence. A shift from calculation skills to other competence and skills.
- A clearer emphasis on problem solving
- A more application oriented mathematics.
- More frequent cross curriculum teaching.

3.1 Some new ways in the exam situation

Based on this recognition we started our recent project. Task was to investigate the consequences of the following models of examining the students learning:

Model 1: "A year´s-time for written exams"

In contrast to the Austrian National Curriculum which prescribes a certain number of hourly written exams per year, our model allows the teachers to test a total of 250 minutes in the academic year, thus permitting them to assess their students in two different ways:

- **Shorter tests** (15 to 30 minutes) to examine certain fundamental competence like calculation competence, visualization competence or also abilities of using the available CAS. In some tests the use of any electronic tool, especially CAS, was forbidden, in others it was allowed.
- **Problem-solving-examinations** (50 to 120 minutes) measure the competence of problem solving with more application-oriented examples, with more open questions, with more emphasis on argumentation, reasoning or interpreting. During these examinations students mostly are allowed to use their learning media like their math school books or their exercise books.

Both the model and the content of the tests were influenced by our discussion about fundamental mathematical competence:

Thesis 7:

The fundamental competence examined by short tests is the basis and the prerequisite for the most important task of mathematics, the solving of problems.

Significant for this model is the idea of the **two phases**:

- At first building the foundation by focussing on a certain mathematical fundamental competence like an algebraic competence and then
- in a second phase using several fundamental competence for problem solving.

The separation of the two sorts of examinations strengthens the "two phase learning model".

I would like to stress my statement that in a learning phase which is focussed on algebraic calculation skills I expect higher abilities than those we demanded as a "long-term competence" in our paper.

Model 2: "Project work"

A certain number of the classic written exams are substituted by projects which are partly done during the lessons but the larger part of the work the students have to do at home. In some classes every single student has to deal with his special theme, in others the mission of the project work is given to a group of students. This way can be especially observed in larger classes because the presentation of the project work of every individual student of the class costs a lot of time.

The content of such project work is themes which allow students to apply formerly learned contents but there are also spheres where the student has to deal independently with new problems or contents. Students have to produce term papers focussing on their theme which are offered to the other students of the class and they have also to carry out a presentation of their results.

The performance appraisal takes place in two ways:

- Observation of the learning process: The independent activity, the ideas of the student, the necessary inputs of the teacher.
- Assessment of the results: The quality of the term papers, the quality of the presentation, the competence during the discussion about the results.

The advantage of this model of examination is that the learning process and the phase of assessment are not separated. Examination is not a singular event which often causes a lot of stress for the students and the result of which often depends on the momentary state of mind of the students. Anyway this model encourages not only the mathematical competence, it also strengthens the other key qualifications, like methodological competence, social competence and personal competence.

Another advantage is that this model allows an inner differentiation which is not possible when using common written tests: More gifted students can work on more demanding problems than not so gifted students

If we could increase the number of such models of learning and examining, the subject mathematics would make a much better contribution to the necessary general education in the age of information technology and the age of life-long learning.

Model 3: Cross curriculum tests

One of the main tasks of the school of the future is a greater emphasis on training of networked_thinking. One sort of exams within the final exam, the Matura, is an oral exam connecting two subjects. Only a few students choose this approach, because before the final exam they cannot experience very often cross curriculum phases of learning and a cross curriculum test until now is not planned.

In comparison with traditional classes in our CAS-classes we watch a growing importance of cross curriculum phases and therefore it was reasonable to consider this fact in the exam situation.

Using the possibilities of the TI-92, especially CBR and CBL, a connection of mathematics and science is obvious. The questions of such cross curriculum tests are dealing with both subjects the results are assessed for the grades of both subjects.

Model 4: “Written group tests”

The use of the tool CAS causes a growing frequency of cooperative learning phases. Using the new expert, the CAS, students more often are working in pairs or groups often sharing their tasks, discussing mathematical problems, exploring mathematical themes together. This cooperative way of learning needs a suitable method of measuring students abilities and competence.

But as long as every student gets his individual grade it is absolutely necessary not only to assess the group-competence, the individual competence must be the central competence for coming to a valid grade. One rule in our model 4 is that a passing grade is only available for a student if neither the individual competence nor the group competence is negative.

Using the project method as a didactic concept a process-oriented controlling of students´ learning is obvious. Not so obvious for me is the product-oriented, singular written group test.

Two teachers decided to investigate the consequence of “written group tests”
Until now I have not received the results but concerning the method I am skeptical because one rule they have is that only pairs of students with the same or similar grades are allowed to work together. This reminds me of the Indian “caste system”. A not so gifted student has no chance to work together with a talented student during the test.

3.2 Some examples

3.2.1 Examples of short tests examining certain fundamental competence:

As I mentioned before fundamental competence is more than calculating skills. Those tests shall assess fundamental mathematical competence like algebraic competence but also the competence of applying certain heuristic strategies like argumenting, reasoning a.s.o.

The first two examples are complete tests, the following are questions suitable to testing the fundamental algebraic competence of chapter two.

Example 3.1: Mag. Ingrid Schirmer-Saneff, 9th grade Gymnasium Berndorf

Short test: 30 minutes, without using TI 92

Goals: Algebraic competence:

- Visualization competence
- Calculation competence
- Numerical competence

1) Visualise the following sets in a coordinate system:

a.) $\{(x,y) \in R^2 | 0 \le x \le 3 \wedge y = 1\}$

b.) $\{(x,y) \in R^2 | 3 \le x \le 7 \wedge -4 \le y \le -2\}$

2) Solve the unequation in $\mathbb{R}$: name the used equivalence transformations

$$\frac{6y+5}{7} - 4 < 4 - 4y$$

3) Solve the following equation:

$$\frac{1}{R_2} = \frac{1}{R} - \frac{1}{R_1} \quad ; \quad R_1 = ?$$

4) Transform into decimals

a.) $4{,}7 * 10^9$

b.) $8{,}59 * 10^{-10}$

Points per example	grades
1.) 8	24 - 23 : grade 1
2.) 6	22 - 20 : grade 2
3.) 6	19 - 15 : grade 3
4.) 4	14 - 12 : grade 4
------	below 12 : grade 5 (negativ)
24	

Example 3.2: Mag. Ingrid Schirmer-Saneff, 9th grade Gymnasium Berndorf

Short test: 20 minutes, without using the TI 92

Goals: Cross curriculum competence

The themes of students´ presentations of their project work were repeated

Some of these themes which students worked (single work):

- Something is stretched
- We explore graphs of kinematics (velocity, acceleration, a.s.o)
- A ball is jumping
- Accidents: Elastic and non elastic collisions

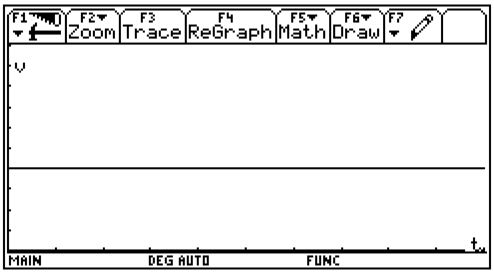
The results are assessed for both subjects, mathematics and science

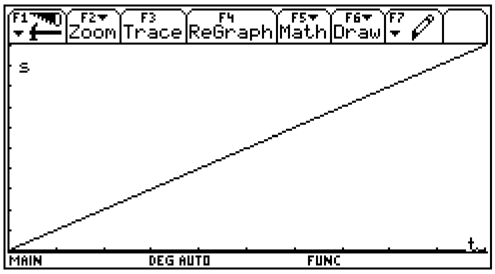
Points: 1.) 2   2.) 3   3.) 4   4.) 4   5.) 3   6.) 3   7.) 5   max 24

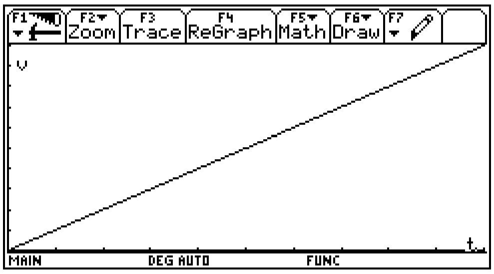
1.) The slope of the function: The stretching of a coil spring with respect to the number of the weight-pieces: The slope of the function informs about

- The number of the weight pieces
- Construction and material of the screwspring
- Stretching of the screwspring
- ....

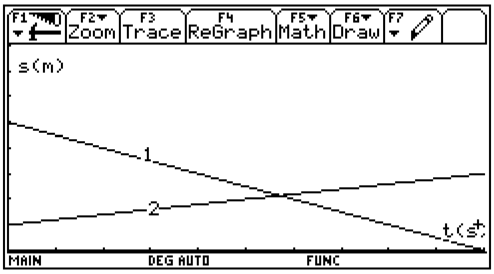
2.)A car is moving with constant velocity. What graph describes this way of <u>movement?</u>





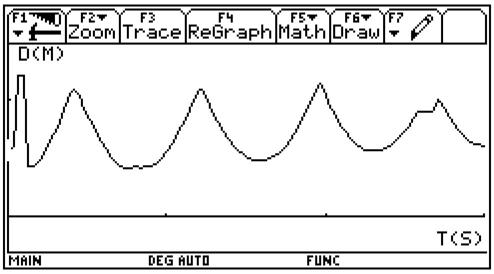


3.) 2 Pupils are approaching each other



The first pupil starts in a distance of ......meters far from the measuring device, the second pupil ......meters. The meeting place is reached after ......seconds in a distance of ......m far from the measuring device. The ...... pupil is walking more slowly than the ......pupil.

4.) A ball is jumping....



For this shot the CBL was situated

- At the starting heights of the ball
- At the floor
- Above the starting heights of the ball

The velocity of the ball is zero if ....

You can recognize the points where the velocity is very high because the curve at this point is

- very flat
- very steep
- horizontal

5.) A ball is falling down. The velocity between dropping and the impact on the floor

- becomes larger and larger
- becomes slower
- is constant

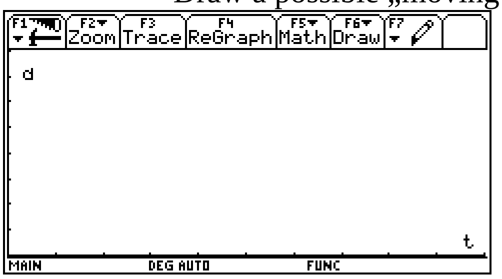
The way of the ball is a

- linear increasing
- quadratic
- linear decreasing

function of the time

6.) Passing of two cyclists

Draw a possible „moving story into the following diagram



7.)The real loss of value of a car

- remains the same every year
- is larger during the first years but diminishes with time
- is growing during the years

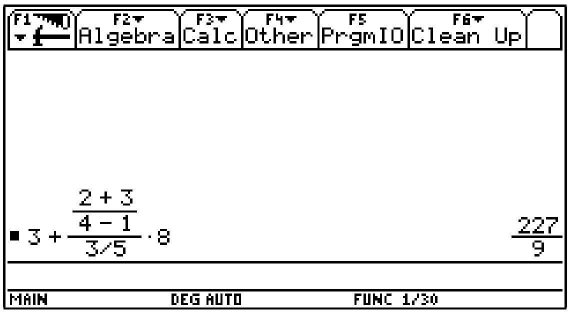
If a car is deducted from taxes for ....years, then the value of the car after three years is.................. of the initial value. This is known as.............

Example 3.3: Mag. Christian Hochfelsner, 7th grade Gymnasium Stockerau

Goal: Structure recognition:

Usage of the TI 92 is allowed

Which keys were activated for the input of this term?



Example 3.4: Mag. Josef Böhm, 6th grade Handelsakademie St. Pölten

Goals: Calculation competence, competence of using the tool TI 92

Usage of the TI 92 is allowed

Calculate with the TI 92 and justify the result: $\dfrac{(12a^2\,b^3\,c^3)^4}{(18\,a^3\,b^2\,c)^3}$

Explain the result of the calculator $\dfrac{x\sqrt{y}+y\sqrt{x}}{\sqrt{y}+\sqrt{x}} =$

Example 3.5: Mag. Hermine Rögner, 9th grade Oberstufengymnasium St. Pölten

Goal: Calculation competence, competence of structure recognition

No use of the TI 92 is allowed

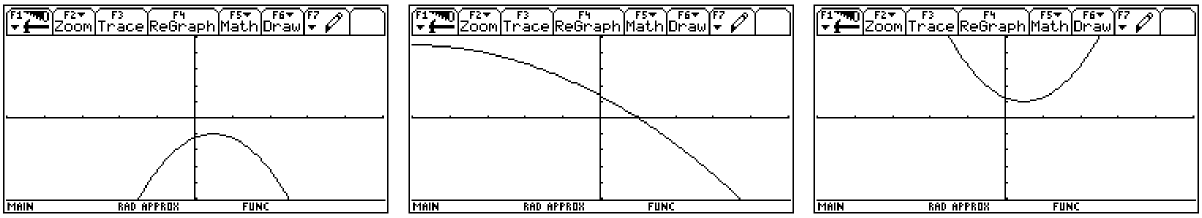
Simplify $\left(\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}\right)\cdot\left(a^2-b^2\right)=$

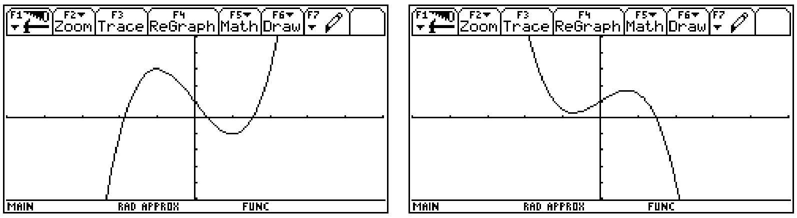
Divide the two fractions and simplify: $\dfrac{3s-r}{25s^2-t^2}:\dfrac{r^2-9s^2}{5s+t}=$

Example 3.6: Dr. Hildegard Urban Woldron, Gymnasium Preßbaum

Goal: Visualisation competence

No use of the TI 92 is allowed





Which of the graphs have the following qualities:?

f'(0)>0 or f'(1)<0 or f''(x) is always negativ.

3.2.2 Problem-solving-exams

The single fundamental competence can be seen as modules which are the necessary building stones for solving problems. After having acquired such a competence the belonging skills e.g. calculation skills can be processed by the tool CAS. The durable, long-term competence is the relational understanding, the competence of planing and deciding a suitable method or algorithm.

Actually the tool CAS makes the solving of interesting, realistic problems at first possible. The separate examination of the fundamental competence and the possibility of using all learning media including instructions about these competence makes the significance of these building stones conscious for the students. Therefore they are able to concentrate their activities in the problem solving strategies.

The following example is typical of a problem solving exam:

Example: Mag. Martin Dangl, Gymnasium Waidhofen/Thaya

Goal: Problem solving

Necessary fundamental competence:

- Finding a formula
- Recognizing structures
- Testing
- Calculating
- Visualizing
- Working with modules (the used formula can be seen as modules)
- Techniques necessary for the used CAS

1500 fish are released into a river. The annual growth rate of the fish population is 35%. This growth of the fish population naturally depends on the amount of fish caught annually and should be discussed using the following three models.

Describe each model by using the appropriate difference equation. Graph the three functions with respect to the time for the first twelve years and describe shortly in words the differences of the three systems with respect to the time..

a) **Model A:** The number of fish caught per year remains constant at 400
   - Solve the difference equation for this model . (Derive an explicit representation of the population number x_n from the difference equation)
   - Calculate the number of years which it takes for the population number to exceed 100 ,000.
   - Calculate how many fish need to be caught annually if no fish are to be left at the end of twelve years.

b) **Model B:** Beginning with a fish population of 400, the amount of fish caught annually should increase by 10% per year.
   - Calculate how many fish in total are caught in the first 12 years.

c) **Model C:** Determine how many fish should be caught annually so that the following conditions are fulfilled:
   (1) The amount of fish caught annually at the beginning should be greater than that in model B.
   (2) The amount of fish caught annually should grow at a constant percentage rate of p.
   (3) The population should grow continuously during the first 12 years.
   (4) A long term unlimited increase in the population must be prevented.

4. Final conclusion

Final results of our running Austrian CAS project III can be expected at the beginning of the next year. In addition to the observations of the teachers and our investigations the center for school evaluation is carrying out an external evaluation.

Not only does the exam situation influence the students´ learning behavior – also the learning process itself influences the exam situation – we watch a closed loop: The idea of our research project was influenced by the observations of the learning process and now a year later we receive a feedback from the behavior in the exam situation to the motivation and strategies in the learning process.

Samour Papert had a dream: Just as you learn a foreign language best in a country where it is spoken you would learn the language of mathematics best in a "mathematic land" and he was sure that his Logo-learning environment would be such a "Mathematic Land". Years later we can say Logo is not the "holy land"

We did not find such a "holy land" when working with CAS, not for the pupils and not for the teachers, but an interesting learning environment for a

- **more meaningful,**
- **more interesting,**
- **and more future oriented**

Mathematics Education

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