Solving Third Degree Polynomial Equations

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Abstract

Starting with some concrete examples, we use TI-Nspire CX CAS to solve third degree polynomial equations with real coefficients. Then we compare the answers obtained to the ones provided by other CAS.

Using Cardano's method or François Viète's formulas, we create a homemade function allowing Nspire to produce clear and compact answers for the solutions of a third degree polynomial equation: "Everything should be made as simple as possible, but not simpler".

Table of contents

Two examples showing CAS imperfections – first example

Consider the following third degree polynomial function

$$f_1(x) = x^3 + 3x + 1$$

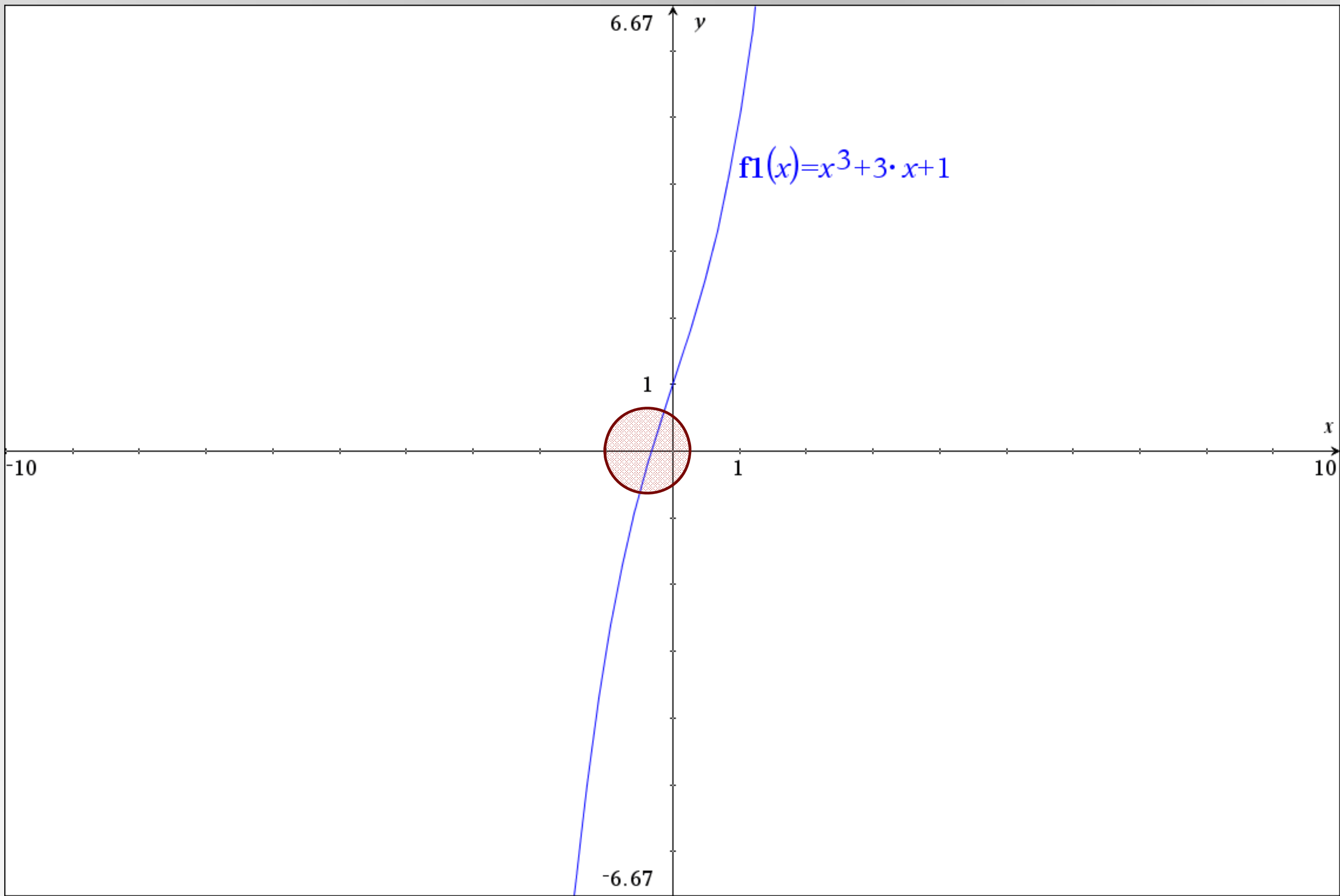
Since :

a. the derivative of the function is strictly positive ;
b. $f_1(-1) = -3$ and $f_1(0) = 1$;

we infer that the function has a single real root located between -1 and 0.

Let's look at the graph of the function to confirm this.

Two examples showing CAS imperfections – first example



Two examples showing CAS imperfections – first example

Let's compare the solutions we obtain when we solve the equation

$$f_1(x) = x^3 + 3x + 1 = 0$$

using 4 different CAS:

- Maple™,
- Mathematica™,
- TI-Nspire™ and
- Derive™.

Two examples showing CAS imperfections – first example

With Maple™:

$$-\frac{1}{2}\left(4+4\sqrt{5}\right)^{1/3}+\frac{2}{\left(4+4\sqrt{5}\right)^{1/3}}$$

$$\frac{1}{4}\left(4+4\sqrt{5}\right)^{1/3}-\frac{1}{\left(4+4\sqrt{5}\right)^{1/3}}+\frac{1}{2}\mathrm{I}\sqrt{3}\left(-\frac{1}{2}\left(4+4\sqrt{5}\right)^{1/3}-\frac{2}{\left(4+4\sqrt{5}\right)^{1/3}}\right)$$

$$\frac{1}{4}\left(4+4\sqrt{5}\right)^{1/3}-\frac{1}{\left(4+4\sqrt{5}\right)^{1/3}}-\frac{1}{2}\mathrm{I}\sqrt{3}\left(-\frac{1}{2}\left(4+4\sqrt{5}\right)^{1/3}-\frac{2}{\left(4+4\sqrt{5}\right)^{1/3}}\right)$$

Two examples showing CAS imperfections – first example

With Mathematica™:

$$-\left(\frac{2}{-1+\sqrt{5}}\right)^{1/3}+\left(\frac{1}{2}\left(-1+\sqrt{5}\right)\right)^{1/3}$$

$$-\frac{1}{2}\left(1+\mathrm{i}\sqrt{3}\right)\left(\frac{1}{2}\left(-1+\sqrt{5}\right)\right)^{1/3}+\frac{1-\mathrm{i}\sqrt{3}}{2^{2/3}\left(-1+\sqrt{5}\right)^{1/3}}$$

$$-\frac{1}{2}\left(1-\mathrm{i}\sqrt{3}\right)\left(\frac{1}{2}\left(-1+\sqrt{5}\right)\right)^{1/3}+\frac{1-\mathrm{i}\sqrt{3}}{2^{2/3}\left(-1+\sqrt{5}\right)^{1/3}}$$

Two examples showing CAS imperfections – first example

With TI-Nspire™:

$$\frac{-2\left(\sqrt{5}+3\right)^{1/3}}{\left(2\left(\sqrt{5}+3\right)\right)^{2/3}+2\left(\sqrt{5}+3\right)^{1/3}+2\cdot 2^{1/3}}$$

$$\frac{\left(\sqrt{5}+1\right)^{2/3}}{\left(\sqrt{5}+1\right)^{4/3}\cdot 2^{1/3}+2\left(\sqrt{5}+1\right)^{2/3}+2\cdot 2^{2/3}}+\frac{\left(2\left(\sqrt{5}-1\right)\right)^{1/3}\cdot\left(\left(\sqrt{5}+1\right)^{2/3}\cdot 2^{1/3}+2\right)\cdot 4^{2/3}\cdot\sqrt{3}}{16}\mathrm{i}$$

$$\frac{\left(\sqrt{5}+1\right)^{2/3}}{\left(\sqrt{5}+1\right)^{4/3}\cdot 2^{1/3}+2\left(\sqrt{5}+1\right)^{2/3}+2\cdot 2^{2/3}}-\frac{\left(2\left(\sqrt{5}-1\right)\right)^{1/3}\cdot\left(\left(\sqrt{5}+1\right)^{2/3}\cdot 2^{1/3}+2\right)\cdot 4^{2/3}\cdot\sqrt{3}}{16}\mathrm{i}$$

Two examples showing CAS imperfections – first example

With Derive™ (notice the simplicity of these solutions):

$$\frac{(4\sqrt{5}-4)^{1/3}}{2}+\frac{(4\sqrt{5}+4)^{1/3}}{2}$$

$$-\frac{(4\sqrt{5}-4)^{1/3}}{4}+\frac{(4\sqrt{5}+4)^{1/3}}{4}+\mathrm{i}\left(\frac{\sqrt{3}(4\sqrt{5}-4)^{1/3}}{4}+\frac{\sqrt{3}(4\sqrt{5}+4)^{1/3}}{4}\right)$$

$$-\frac{(4\sqrt{5}-4)^{1/3}}{4}+\frac{(4\sqrt{5}+4)^{1/3}}{4}-\mathrm{i}\left(\frac{\sqrt{3}(4\sqrt{5}-4)^{1/3}}{4}+\frac{\sqrt{3}(4\sqrt{5}+4)^{1/3}}{4}\right)$$

Two examples showing CAS imperfections – second example

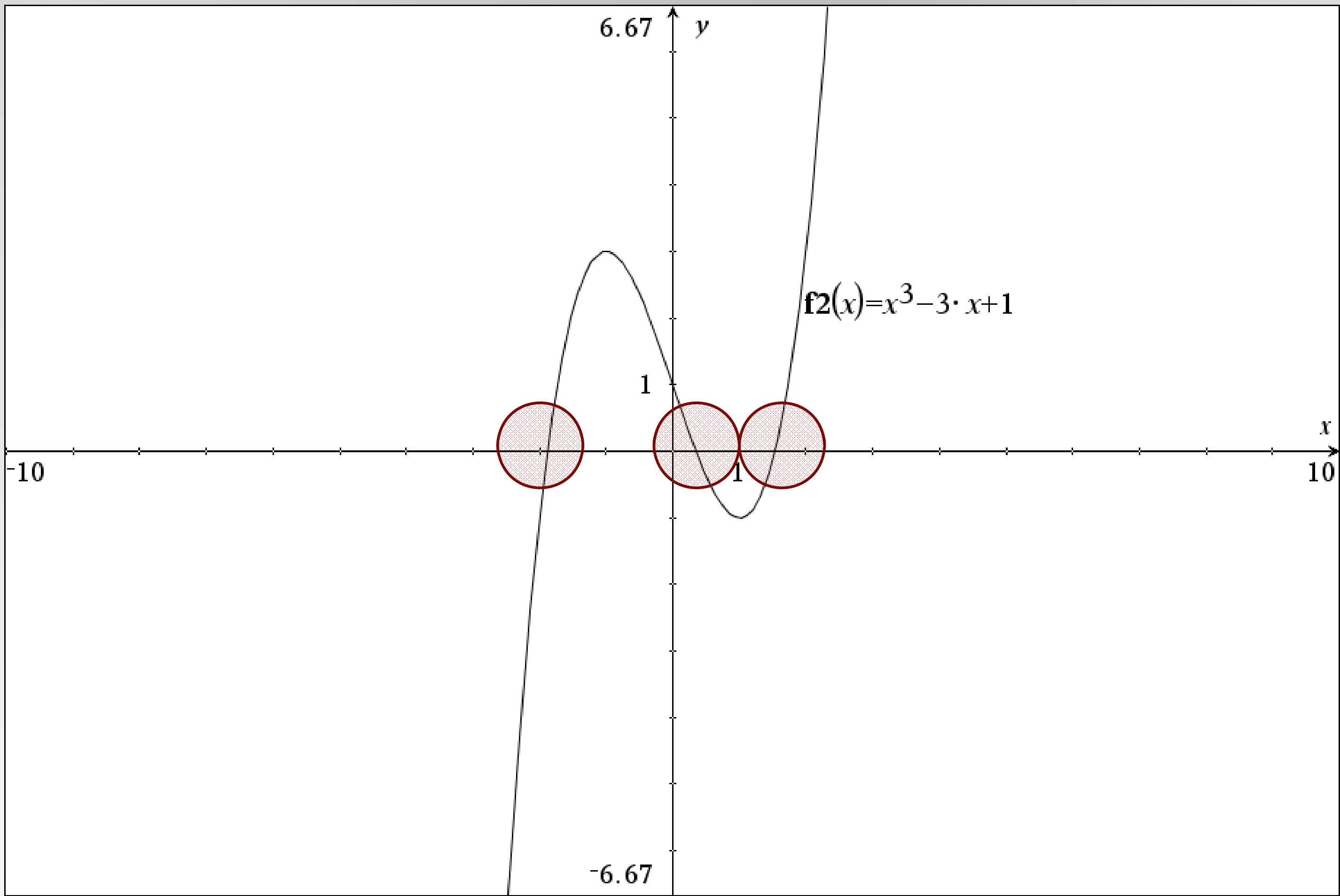
Consider now this other simple third degree polynomial function

$$f_2(x) = x^3 - 3x + 1$$

Having 2 critical points, we may expect this function to have 3 real roots.

Let’s look at the graph of the function to confirm this.

Two examples showing CAS imperfections – second example



Two examples showing CAS imperfections – second example

Again, let’s compare the solutions we obtain when we solve the equation

$$f_2(x) = x^3 - 3x + 1 = 0$$

using 4 different CAS:

- Maple™,
- Mathematica™,
- TI-Nspire™ and
- Derive™.

Two examples showing CAS imperfections – second example

With Maple™ (it's hard to believe these are real values)

$$\frac{1}{2}\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}+\frac{2}{\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}}$$

$$-\frac{1}{4}\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}-\frac{1}{\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}}+\frac{1}{2}\mathrm{I}\sqrt{3}\left(\frac{1}{2}\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}-\frac{2}{\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}}\right)$$

$$-\frac{1}{4}\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}-\frac{1}{\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}}-\frac{1}{2}\mathrm{I}\sqrt{3}\left(\frac{1}{2}\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}-\frac{2}{\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}}\right)$$

Two examples showing CAS imperfections – second example

With Mathematica™:

$$\frac{1}{\left(\frac{1}{2}\left(-1+\mathrm{i}\sqrt{3}\right)\right)^{1/3}}+\left(\frac{1}{2}\left(-1+\mathrm{i}\sqrt{3}\right)\right)^{1/3}$$

$$-\frac{1-\mathrm{i}\sqrt{3}}{2^{2/3}\left(-1+\mathrm{i}\sqrt{3}\right)^{1/3}}-\frac{1}{2}\left(\frac{1}{2}\left(-1+\mathrm{i}\sqrt{3}\right)\right)^{1/3}\left(1+\mathrm{i}\sqrt{3}\right)$$

$$-\frac{1}{2}\left(1-\mathrm{i}\sqrt{3}\right)\left(\frac{1}{2}\left(-1+\mathrm{i}\sqrt{3}\right)\right)^{1/3}-\frac{1+\mathrm{i}\sqrt{3}}{2^{2/3}\left(-1+\mathrm{i}\sqrt{3}\right)^{1/3}}$$

Two examples showing CAS imperfections – second example

With TI-Nspire™:

$$-\left(\cos\left(\frac{2\cdot\pi}{9}\right)+\sin\left(\frac{2\cdot\pi}{9}\right)\cdot\sqrt{3}\right)$$

$$\frac{1}{\cos\left(\frac{2\cdot\pi}{9}\right)+\sin\left(\frac{2\cdot\pi}{9}\right)\cdot\sqrt{3}+1}$$

$$\frac{\cos\left(\frac{2\cdot\pi}{9}\right)+\sin\left(\frac{2\cdot\pi}{9}\right)\cdot\sqrt{3}+1}{\cos\left(\frac{2\cdot\pi}{9}\right)+\sin\left(\frac{2\cdot\pi}{9}\right)\cdot\sqrt{3}}$$

Two examples showing CAS imperfections – second example

With Derive™ (again, so much simpler):

$$2 \cdot \cos\left(\frac{2\pi}{9}\right)$$

$$-2 \cdot \cos\left(\frac{\pi}{9}\right)$$

$$2 \cdot \sin\left(\frac{\pi}{18}\right)$$

An algorithm to beautify the solutions

Our goal is to **create a new function for TI-Nspire** that finds the roots of a third degree polynomial function while:

a. avoiding the presence of radicals in the denominators of the roots;

b. avoiding needless occurrence of i in the real roots and using, instead, trig expressions for the roots;

The next slide presents the algorithm we will implement.

An algorithm to beautify the solutions

Choose a 3rd degree function

↓

Get rid of the 2nd degree term

↓

Ascertain the number of real roots

1 real root

Transform into a 6th degree polynomial equation

↓

Transform into a 2nd degree polynomial equation

↓

Select one solution of the previous equation

↓

Compute the 3 cubic roots of this solution

↓

Compute the roots of the original function

3 real roots

Apply a trigonometric substitution

↓

Solve the equation

↓

Compute the roots of the original function

Solving a third degree polynomial equation with 1 real root

Let's use our algorithm to **solve a third degree polynomial equation that possesses a <u>single real root</u>**.

To solve the equation, we will need to accomplish 8 steps, as seen in the algorithm's organigram.

START → 2 → 3 → 4 → 5 → 6 → 7 → 8

Solving a third degree polynomial equation with 1 real root

Step #1 – Pick a 3rd degree polynomial function

SELECT A 3RD DEGREE POLYNOMIAL FUNCTION IN THE FORM OF $f(x) = ax^3 + bx^2 + cx + d$.

Let's apply our algorithm to solve the equation

$$f(x) = x^3 - 4x^2 + 2x - 5 = 0$$

As we will see, this function only possess a single real root.

START → 2 → 3 → 4 → 5 → 6 → 7 → 8

Solving a third degree polynomial equation with 1 real root

Step #2 – Get rid of the 2nd degree term

DEFINE $g(y)$ BY SUBSTITUTING $y - {}^{b}/_{3a}$ FOR x AND THEN DIVIDING BY a.

Since $f(x) = x^3 - 4x^2 + 2x - 5 = 0$, we obtain:

$$g(y) = f\left(y - \frac{-4}{3}\right) = y^3 - \frac{10}{3}y - \frac{191}{27} = 0$$

START → 2 → 3 → 4 → 5 → 6 → 7 → 8

Solving a third degree polynomial equation with 1 real root

Step #3 – Ascertain the number of real roots

REWRITE $g(y)$ IN THE FORM OF $y^3 + 3py - 2q$ WHERE $p = \frac{3ac-b^2}{9a^2}$ AND $q = -\frac{(27a^2d-9abc+2b^3)}{54a^3}$.

Since $g(y) = y^3 - \frac{10}{3}y - \frac{191}{27} = 0$, we find that $p = \frac{-10}{9}$ and $q = \frac{191}{54}$.

START → 2 → 3 → 4 → 5 → 6 → 7 → 8

Solving a third degree polynomial equation with 1 real root

Step #3 – Ascertain the number of real roots

IF $q^2 + p^3 > 0$, THERE IS A SINGLE REAL SOLUTION. OTHERWISE, THERE ARE 3 REAL SOLUTIONS.

Since $p = \frac{-10}{9}$ and $q = \frac{191}{54}$, $q^2 + p^3 = \frac{401}{36} > 0$.

We conclude that **our equation only has a single real solution**.

START → 2 → 3 → 4 → 5 → 6 → 7 → 8

Solving a third degree polynomial equation with 1 real root

Step #4 – Transform the equation into a 6th degree polynomial equation

DEFINE $h(u) = u^6 - 2qu^3 - p^3$ BY SUBSTITUTING $u - {}^{p}/_{u}$ FOR y IN $g(y)$ AND THEN MULTIPLYING BY u^3.

Since $g(y) = y^3 - \frac{10}{3}y - \frac{191}{27} = 0$, we find:

$$h(u) = u^6 - \frac{191}{27}u^3 + \frac{1000}{729} = 0$$

START → 2 → 3 → 4 → 5 → 6 → 7 → 8

Solving a third degree polynomial equation with 1 real root

Step #5 – Transform the equation into a 2nd degree polynomial equation

DEFINE $k(z) = z^2 - 2qz - p^3$ BY SUBSTITUTING z FOR u^3 IN $h(u)$.

Since $h(u) = u^6 - \frac{191}{27}u^3 + \frac{1000}{729} = 0$, we find:

$$k(z) = z^2 - \frac{191}{27}z - \frac{1000}{729} = 0$$

START → 2 → 3 → 4 → **5** → 6 → 7 → 8

Solving a third degree polynomial equation with 1 real root

Step #6 – Select 1 of the 2 solutions of the 2nd degree polynomial equation

SOLVE THE EQUATION $k(z) = z^2 - 2qz - p^3 = 0$ AND SELECT THE SOLUTION $\sqrt{p^3 + q^2} + q$.

After solving $k(z) = z^2 - \frac{191}{27}z - \frac{1000}{729} = 0$, we select the solution $\frac{9 \cdot \sqrt{401} + 191}{54}$.

START → 2 → 3 → 4 → 5 → 6 → 7 → 8

Solving a third degree polynomial equation with 1 real root

Step #7 – Compute the cubic roots of the chosen solution

SINCE $z = \sqrt{p^3 + q^2} + q$ AND $z = u^3$, SOLVE THE EQUATION $u^3 = \sqrt{p^3 + q^2} + q$.

Solving $u^3 = \frac{9 \cdot \sqrt{401} + 191}{54}$, we find 3 solutions that we note u_0, u_1 and u_2.

START → 2 → 3 → 4 → 5 → 6 → 7 → 8

Solving a third degree polynomial equation with 1 real root

Step #7 – Compute the cubic roots of the chosen solution

The solutions of the equation $u^3 = \frac{9 \cdot \sqrt{401} + 191}{54}$ are:

$$u_0 = \frac{\left(9 \cdot \sqrt{401} + 191\right)^{\frac{1}{3}} \cdot 2^{\frac{2}{3}}}{6}$$

$$u_1 = \frac{-\left(9 \cdot \sqrt{401} + 191\right)^{\frac{1}{3}} \cdot 2^{\frac{2}{3}}}{12} + \frac{\left(9 \cdot \sqrt{401} + 191\right)^{\frac{1}{3}} \cdot \sqrt{3} \cdot 2^{\frac{2}{3}}}{12}\mathbf{i}$$

$$u_2 = \frac{-\left(9 \cdot \sqrt{401} + 191\right)^{\frac{1}{3}} \cdot 2^{\frac{2}{3}}}{12} - \frac{\left(9 \cdot \sqrt{401} + 191\right)^{\frac{1}{3}} \cdot \sqrt{3} \cdot 2^{\frac{2}{3}}}{12}\mathbf{i}$$

START → 2 → 3 → 4 → 5 → 6 → 7 → 8

Solving a third degree polynomial equation with 1 real root

Step #8 – Compute the roots of the original function

USING THE 3 ROOTS OF THE FUNCTION $h(u)$, COMPUTE THE 3 ROOTS OF THE FUNCTION $g(y)$.

Since $y = u - {}^{p}/_{u}$, we compute the 3 roots of $g(y)$:

$$y_0 = u_0 - {}^{p}/_{u_0}$$

$$y_1 = u_1 - {}^{p}/_{u_1}$$

$$y_2 = u_2 - {}^{p}/_{u_2}$$

START → 2 → 3 → 4 → 5 → 6 → 7 → 8

Solving a third degree polynomial equation with 1 real root

Step #8 – Compute the roots of the original function

USING THE 3 ROOTS OF THE FUNCTION $g(y)$, COMPUTE THE 3 ROOTS OF THE FUNCTION $f(x)$.

Since $x = y - {}^{b}/_{3a}$, we compute the 3 roots of $f(x)$:

$$x_0 = y_0 - {}^{-4}/_{3} = y_0 + {}^{4}/_{3}$$

$$x_1 = y_1 - {}^{-4}/_{3} = y_1 + {}^{4}/_{3}$$

$$x_2 = y_2 - {}^{-4}/_{3} = y_2 + {}^{4}/_{3}$$

START → 2 → 3 → 4 → 5 → 6 → 7 → 8

Solving a third degree polynomial equation with 3 real roots

Let's use our algorithm to **solve a third degree polynomial equation that possesses 3 real roots**.

To solve the equation, we will need to accomplish only 6 steps, as seen in the algorithm's organigram.

START → **2** → **3** → **4** → **5** → **6**

Solving a third degree polynomial equation with 3 real roots

Step #1 – Pick a 3rd degree polynomial function

SELECT A 3RD DEGREE POLYNOMIAL FUNCTION IN THE FORM OF $f(x) = ax^3 + bx^2 + cx + d$.

Let's apply our algorithm again to solve the equation

$$f(x) = x^3 + 4x^2 + x - 1 = 0$$

As we will see, this function only possess 3 real roots.

START → **2** → **3** → **4** → **5** → **6**

Solving a third degree polynomial equation with 3 real roots

Step #2 – Get rid of the 2nd degree term

DEFINE $g(y)$ BY SUBSTITUTING $y - {}^{b}/_{3a}$ FOR x AND THEN DIVIDING BY a.

Since $f(x) = x^3 + 4x^2 + x - 1 = 0$, we obtain:

$$g(y) = f\left(y - \frac{4}{3}\right) = y^3 - \frac{13}{3}y + \frac{65}{27} = 0$$

START → 2 → 3 → 4 → 5 → 6

Solving a third degree polynomial equation with 3 real roots

Step #3 – Find the number of real roots

REWRITE $g(y)$ IN THE FORM OF $y^3 + 3py - 2q$ WHERE $p = \frac{3ac-b^2}{9a^2}$ AND $q = -\frac{(27a^2d-9abc+2b^3)}{54a^3}$.

Since $g(y) = y^3 - \frac{13}{3}y + \frac{65}{27} = 0$, we find that $p = \frac{-13}{9}$ and $q = \frac{-65}{54}$.

START → 2 → 3 → 4 → 5 → 6

Solving a third degree polynomial equation with 3 real roots

Step #3 – Ascertain the number of real roots

If $q^2 + p^3 > 0$, there is single real solution. Otherwise, there are 3 real solutions.

Since $p = \frac{-13}{9}$ and $q = \frac{-65}{54}$, $q^2 + p^3 = \frac{-169}{108} < 0$.

We conclude that **our equation has 3 real solutions**.

START → **2** → **3** → **4** → **5** → **6**

Solving a third degree polynomial equation with 3 real roots

Step #4 – Apply a substitution

DEFINE $h(\theta) = 2p\sqrt{-p} \cdot \sin(3\theta) - 2q$ BY SUBSTITUTING $2\sqrt{-p} \cdot \sin(\theta)$ FOR y IN $g(y)$, WHERE $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Since $g(y) = y^3 - \frac{13}{3}y + \frac{65}{27} = 0$, we find:

$$h(\theta) = \frac{65}{27} - \frac{26\sin(3\theta) \cdot \sqrt{13}}{27} = 0$$

START → 2 → 3 → **4** → 5 → 6

Solving a third degree polynomial equation with 3 real roots

Step #5 – Solve the equation

COMPUTE THE 3 ROOTS OF $h(\theta) = 2p\sqrt{-p} \cdot \sin(3\theta) - 2q$:

$\frac{\sin^{-1}\left(\frac{q}{p\sqrt{-p}}\right)}{3}$, $\frac{\pi - \sin^{-1}\left(\frac{q}{p\sqrt{-p}}\right)}{3}$ AND $-\left(\frac{\pi + \sin^{-1}\left(\frac{q}{p\sqrt{-p}}\right)}{3}\right)$.

And so we find:

$$\theta_0 = \frac{\sin^{-1}\left(\frac{5\sqrt{13}}{26}\right)}{3} \qquad \theta_1 = \frac{\pi - \sin^{-1}\left(\frac{5\sqrt{13}}{26}\right)}{3} \qquad \theta_2 = -\left(\frac{\pi + \sin^{-1}\left(\frac{5\sqrt{13}}{26}\right)}{3}\right)$$

START → 2 → 3 → 4 → **5** → 6

Solving a third degree polynomial equation with 3 real roots

Step #6 – Compute the roots of the original function

USING THE 3 ROOTS OF THE FUNCTION $h(\theta)$, COMPUTE THE 3 ROOTS OF THE FUNCTION $g(y)$.

Since $y = 2\sqrt{-p} \cdot \sin(\theta)$, we compute the 3 roots of $g(y)$:

$$y_0 = 2\sqrt{-p} \cdot \sin(\theta_0)$$
$$y_1 = 2\sqrt{-p} \cdot \sin(\theta_1)$$
$$y_2 = 2\sqrt{-p} \cdot \sin(\theta_2)$$

START → 2 → 3 → 4 → 5 → 6

Solving a third degree polynomial equation with 3 real roots

Step #6 – Compute the roots of the original function

USING THE 3 ROOTS OF THE FUNCTION $g(y)$, COMPUTE THE 3 ROOTS OF THE FUNCTION $f(x)$.

Since $x = y - \frac{4}{3}$, we compute the 3 roots of $f(x)$:

$$x_0 = y_0 - {}^4\!/_3 = 2\sqrt{-p} \cdot \sin(\theta_0) - {}^4\!/_3$$

$$x_1 = y_1 - {}^4\!/_3 = 2\sqrt{-p} \cdot \sin(\theta_1) - {}^4\!/_3$$

$$x_2 = y_2 - {}^4\!/_3 = 2\sqrt{-p} \cdot \sin(\theta_2) - {}^4\!/_3$$

START → 2 → 3 → 4 → 5 → 6

Implementation of a new TI-Nspire function

This new function has been implemented by Michel Beaudin and can be downloaded. Go to

https://cours.etsmtl.ca/seg/mbeaudin/

And save the TI-Nspire CX CAS library “Kit_ETS_MB” into MyLib. The name of the function is “compact_cubic”.

Live examples: on the TI-Nspire CX CAS file.

Conclusion

Teaching mathematics using CAS is, as far as we are concerned, an excellent opportunity for mathematics teachers to stay enthusiastic even though the curriculum has been quite the same for the several past years!